LAGRANGIAN MECHANICS
LECTURE NOTES
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Contents

The present document contains lecture notes for an elective course at Bern University of Applied Sciences. The notes are meant as a complementary aid for students and no claim to completeness is made.

1. Newtonian Mechanics

1.1. **Single Particle.**

1.1.1. *Kinematics.* We call a body whose extension can be neglected a *particle* and denote by $\boldsymbol{r}$ its position relative to a *reference frame*. This means that the three components of the vector $\boldsymbol{r}$ are cartesian coordinates. We denote them by x, y, z. The *motion* of a particle is then given by the map

$$t \mapsto \boldsymbol{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} , \tag{1}$$

where the parameter t is time. The *velocity* $\boldsymbol{v}(t)$ and *acceleration* $\boldsymbol{a}(t)$ are given by

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t) \,, \qquad \boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{r}}(t) \,, \tag{2}$$

where we use the notation

$$\dot{\boldsymbol{r}}(t) = \frac{d\boldsymbol{r}}{dt}(t) \tag{3}$$

and the derivative of a vector is given by its component-wise derivative

$$\dot{\boldsymbol{r}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} . \tag{4}$$

1.1.2. *Newtons First Law of Motion.* Newtons first law states that:

A body remains at rest or in uniform motion unless acted upon by a force.

In this statement we identify the body with a particle and by being at rest or in uniform motion it is meant that either $\boldsymbol{v} = \boldsymbol{0}$ or $\boldsymbol{v} = \boldsymbol{const} \neq \boldsymbol{0}$, i.e.

$$\boldsymbol{a} = \boldsymbol{0} \,. \tag{5}$$

The meaning of the term *force* is only given later through the statement of Newtons second law of motion (see 1.1.3) but it originates from some kind of interaction involving the particle. We call a particle which does not interact a *free* particle and through this notion reformulate Newtons first law of motion as:

There exist reference frames, relative to which a free particle either rests or moves with constant velocity.

These reference frames are called *inertial frames*. I.e. Newtons first law of motion postulates the existence of inertial frames. A good example for an inertial frame is a frame which is at rest relative to the fixed stars.

There exist infinitely many inertial frames, related to each other by coordinate transformations of the form

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \tilde{\boldsymbol{r}} \,. \tag{6}$$

Examples:

(i) $\tilde{\boldsymbol{r}} = \boldsymbol{r} + \boldsymbol{b}$ with $\boldsymbol{b}$ a constant vector is a translation.
(ii) $\tilde{\boldsymbol{r}} = R\boldsymbol{r}$ with R a constant rotation matrix R is a rotation.
(iii) $\tilde{\boldsymbol{r}} = \boldsymbol{r} + t\boldsymbol{c}$ with $\boldsymbol{c}$ a constant vector is a motion of the $\tilde{x}$-$\tilde{y}$-$\tilde{z}$-frame relative to the x-y-z-frame with velocity $\boldsymbol{c}$. Such a transformation is called a *Galilean transformation*.
(iv) $t \mapsto \tilde{t} = t + d$ with d a constant is a time translation.

Remarks:

(i) For each of the above examples from $\ddot{\boldsymbol{r}} = \boldsymbol{0}$ follows that $\ddot{\tilde{\boldsymbol{r}}} = \boldsymbol{0}$.
(ii) The transformation

$$\boldsymbol{r} \mapsto \tilde{\boldsymbol{r}} = R\boldsymbol{r} + t\boldsymbol{c} + \boldsymbol{a} \tag{7}$$

$$t \mapsto \tilde{t} \ = t + d \tag{8}$$

is called a *general Galilean transformation*.

1.1.3. *Newtons Second Law of Motion.* Newtons second law of motion states that:

The time rate of change of momentum equals the force acting on the particle.

The corresponding mathematical statement is

$$\dot{\boldsymbol{p}} = \boldsymbol{F} \,, \tag{9}$$

where we define

$$\boldsymbol{p} := m\boldsymbol{v} \tag{10}$$

as the (linear) *momentum*. Here m is the *mass*, a (scalar) property of the particle measuring the amount of matter the particle contains. Newtons second law of motion then gives the meaning of the force $\boldsymbol{F}$ acting on the particle, which is a vector.

Remarks:

(i) (9) is a vector equation whose components:

$$F_x = \frac{d}{dt}(mv_x) \qquad F_y = \frac{d}{dt}(mv_y) \,, \qquad F_z = \frac{d}{dt}(mv_z) \tag{11}$$

are components relative to an inertial frame.
(ii) In general the force acting on a particle depends on the position $\boldsymbol{r}$, the velocity $\dot{\boldsymbol{r}}$ and the time t, i.e. $\boldsymbol{F} = \boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t)$.
(iii) If $\boldsymbol{F}(\boldsymbol{r}, \dot{\boldsymbol{r}}, t)$ is known, (9) is a differential equation for the position $\boldsymbol{r}(t)$ of the particle:

$$m\ddot{\boldsymbol{r}}(t) = \boldsymbol{F}\left(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t), t\right) \,. \tag{12}$$

1.1.4. *Conservation of Linear Momentum.* If the force vanishes, then the momentum is a conserved quantity. I.e.

$$\boldsymbol{F} = 0 \qquad \Rightarrow \qquad \dot{\boldsymbol{p}} = \boldsymbol{0} \tag{13}$$

Proof. The proof of this statement is a direct consequence of Newtons second law. ☐

1.1.5. *Conservation of Angular Momentum.* Let

$$\boldsymbol{L} := \boldsymbol{r} \times \boldsymbol{p}\,, \qquad \boldsymbol{M} := \boldsymbol{r} \times \boldsymbol{F} \tag{14}$$

be the *angular momentum* and the *torque* respectively. We have

$$\begin{aligned}\dot{\boldsymbol{L}} &= \frac{d}{dt}(\boldsymbol{r} \times \boldsymbol{p}) \\ &= \dot{\boldsymbol{r}} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}} \\ &= \boldsymbol{v} \times m\boldsymbol{v} + \boldsymbol{r} \times \boldsymbol{F} \\ &= m\underbrace{\boldsymbol{v} \times \boldsymbol{v}}_{=\boldsymbol{0}} + \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{M}\,.\end{aligned} \tag{15}$$

I.e. the rate of change of angular momentum equals the torque. The conservation of angular momentum

$$\boldsymbol{M} = \boldsymbol{0} \qquad \Rightarrow \qquad \dot{\boldsymbol{L}} = \boldsymbol{0} \tag{16}$$

follows analogous to the conservation of linear momentum.

1.1.6. *Energy.* Let

$$T := \frac{mv^2}{2} \tag{17}$$

be the *kinetic energy*, where $v = |\boldsymbol{v}|$. Assuming that the mass m is constant, we have

$$\frac{dT}{dt} = \frac{m}{2}\frac{d}{dt}(\boldsymbol{v} \cdot \boldsymbol{v}) = \frac{m}{2}(\dot{\boldsymbol{v}} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \dot{\boldsymbol{v}}) = m\boldsymbol{v} \cdot \dot{\boldsymbol{v}} = \boldsymbol{v} \cdot \boldsymbol{F}\,. \tag{18}$$

For an interpretation of this expression we consider a particle at two times t_1, t_2. The change in kinetic energy is

$$T_2 - T_1 = \int_{t_1}^{t_2} \frac{dT}{dt} dt = \int_{t_1}^{t_2} \boldsymbol{F} \cdot \boldsymbol{v}\, dt\,, \tag{19}$$

where we use the notation $T_i = T(t_i)$ and dropped the arguments in the integral for better readability. We rewrite

$$\boldsymbol{F} \cdot \boldsymbol{v}\, dt = \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt}\, dt = \boldsymbol{F} \cdot d\boldsymbol{r}\,. \tag{20}$$

The corresponding finite expression is

$$\boldsymbol{F} \cdot \Delta\boldsymbol{r} = |\boldsymbol{F}||\Delta\boldsymbol{r}|\cos(\theta)\,, \tag{21}$$

where θ is the angle between the force $\boldsymbol{F}$ and the change in position $\Delta\boldsymbol{r}$. I.e. this expression equals the component of $\boldsymbol{F}$ in the direction of $\Delta\boldsymbol{r}$ times the absolute value of the change in position $|\Delta\boldsymbol{r}|$. This is defined as the *work* done by the force $\boldsymbol{F}$ on the particle while displacing it by $\Delta\boldsymbol{r}$. I.e. the difference in kinetic energy equals the work done on the particle.

We call $\boldsymbol{F}$ *conservative* if there exists a function $U(x, y, z)$ such that $\boldsymbol{F} = -\nabla U$.

Remarks:

(i) U is called *potential* or *potential energy.*
(ii) $\nabla U = \nabla U(x, y, z)$ and since $\boldsymbol{F} = -\nabla U$, we have that $\boldsymbol{F}(\boldsymbol{r}(t)) = -\nabla U(\boldsymbol{r}(t))$.
(iii) The potential U is only fixed up to a constant, i.e. $\nabla U = \nabla(U + C)$ for C a constant.

We call $E = T + U$ the *total energy.*

Proposition 1.1. *If $\boldsymbol{F}$ is conservative, then the total energy $E = T + U$ is conserved.*

Proof. From (18) we have

$$\frac{dT}{dt}(t) = \boldsymbol{v}(t) \cdot \boldsymbol{F}(\boldsymbol{r}(t)) = -\boldsymbol{v}(t) \cdot \nabla U(\boldsymbol{r}(t))\,, \tag{22}$$

where we wrote out all the argument. Using the chain rule we have

$$\begin{aligned}\frac{dU}{dt}(t) = \frac{d}{dt}U(\boldsymbol{r}(t)) &= \frac{\partial U}{\partial x}(\boldsymbol{r}(t))\dot{x}(t) + \frac{\partial U}{\partial y}(\boldsymbol{r}(t))\dot{y}(t) + \frac{\partial U}{\partial z}(\boldsymbol{r}(t))\dot{z}(t) \\ &= \nabla U(\boldsymbol{r}(t)) \cdot \dot{\boldsymbol{r}}(t)\end{aligned}$$

$$= \nabla U(\boldsymbol{r}(t)) \cdot \boldsymbol{v}(t)\,, \tag{23}$$

where for the first equal sign we used the short notation $U(t) = U(\boldsymbol{r}(t))$. Putting together (22) and (23) yields

$$\frac{dE}{dt}(t) = \frac{dT}{dt}(t) + \frac{dU}{dt}(t) = 0\,. \tag{24}$$

□

1.2. **System of Particles.** We consider a system of n particles labeled by the index $i = 1, \ldots, n$. The equation of motion for the i'th particle is Newtons law of motion (9) with the force given as the sum of an external force $\boldsymbol{F}_{ie}$ and forces due to interactions of the i'th with the j'th ($j \neq i$) particle $\boldsymbol{F}_{ij}$:

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_{ie} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \boldsymbol{F}_{ij}\,, \tag{25}$$

where $\boldsymbol{p}_i = m_i \boldsymbol{v}_i = m_i \dot{\boldsymbol{r}}_i$ and $\boldsymbol{r}_i$ denotes the position of the i'th particle. In the following we restrict our treatment to a system of $n = 2$ particles, all our results can be generalized to the case of $n > 2$. For $n = 2$, the equations of motion for the two particles are

$$\dot{\boldsymbol{p}}_1 = \boldsymbol{F}_{1e} + \boldsymbol{F}_{12}\,, \tag{26}$$
$$\dot{\boldsymbol{p}}_2 = \boldsymbol{F}_{2e} + \boldsymbol{F}_{21}\,. \tag{27}$$

1.2.1. *Newtons Third Law of Motion.* Newtons third law of motion states that:

> *The forces that two particles exert on each other are equal in magnitude, of opposite direction and parallel to the line joining the two particles.*

The corresponding mathematical statement is

$$\boldsymbol{F}_{12} = -\boldsymbol{F}_{21} \qquad \text{and} \qquad \boldsymbol{F}_{12} \parallel \boldsymbol{r}_2 - \boldsymbol{r}_1\,. \tag{28}$$

1.2.2. *Center of Mass Motion.* Adding equations (26), (27) we obtain

$$\begin{aligned}\dot{\boldsymbol{p}}_1 + \dot{\boldsymbol{p}}_2 &= \boldsymbol{F}_{1e} + \boldsymbol{F}_{2e} + \boldsymbol{F}_{12} + \boldsymbol{F}_{21} \\ &= \boldsymbol{F}_{1e} + \boldsymbol{F}_{2e}\,,\end{aligned} \tag{29}$$

where we made use of the first of (28). Defining the *total momentum* $\boldsymbol{P}$ and the *total external force* $\boldsymbol{F}_e$ by

$$\boldsymbol{P} := \boldsymbol{p}_1 + \boldsymbol{p}_2\,, \tag{30}$$
$$\boldsymbol{F}_e := \boldsymbol{F}_{1e} + \boldsymbol{F}_{2e}\,, \tag{31}$$

respectively, (29) becomes

$$\dot{\boldsymbol{P}} = \boldsymbol{F}_e\,. \tag{32}$$

I.e. if the external forces vanish then the total momentum is conserved, i.e.

$$\boldsymbol{F}_{ie} = \boldsymbol{0} \qquad \Rightarrow \qquad \dot{\boldsymbol{P}} = \boldsymbol{0}\,. \tag{33}$$

With the *total mass* M and the *center of mass* $\boldsymbol{R}$ defined by

$$M := m_1 + m_2\,, \tag{34}$$
$$\boldsymbol{R} := \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{M}\,, \tag{35}$$

respectively, (32) becomes

$$M\ddot{\boldsymbol{R}} = \boldsymbol{F}_e\,, \tag{36}$$

the equation of motion for a body of mass M and position $\boldsymbol{R}$ under the influence of a force $\boldsymbol{F}_e$.

If $\boldsymbol{F}_e = \boldsymbol{0}$, the center of mass follows a uniform motion, i.e.

$$\ddot{\boldsymbol{R}} = \boldsymbol{0}\,. \tag{37}$$

1.2.3. *Angular Momentum.* By taking the cross product of (26), (27) with $\boldsymbol{r}_1$, $\boldsymbol{r}_2$, respectively and adding the resulting equations, it can be shown that (see problem 1.2)

$$\dot{\boldsymbol{L}} = \boldsymbol{M}\,, \tag{38}$$

where the *total angular momentum* $\boldsymbol{L}$ and the *total torque* $\boldsymbol{M}$ are given as

$$\boldsymbol{L} := \boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_1 \times \boldsymbol{p}_1\,, \tag{39}$$

$$\boldsymbol{M} := \boldsymbol{r}_1 \times \boldsymbol{F}_{1e} + \boldsymbol{r}_2 \times \boldsymbol{F}_{2e}\,. \tag{40}$$

I.e. the rate of change of total angular moment equals the total torque. In particular, if the torque vanishes, the total angular momentum is conserved.

1.2.4. *Energy.* The kinetic energy of the two particles is given by

$$T = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2}\,. \tag{41}$$

We have

$$\begin{aligned}\frac{dT}{dt} &= m_1 \boldsymbol{v}_1 \cdot \dot{\boldsymbol{v}}_1 + m_2 \boldsymbol{v}_2 \cdot \dot{\boldsymbol{v}}_2 \\ &= \boldsymbol{v}_1 \cdot \boldsymbol{F}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{F}_2\,,\end{aligned} \tag{42}$$

where

$$\boldsymbol{F}_i = \boldsymbol{F}_{ie} + \boldsymbol{F}_{ij}\,. \tag{43}$$

Proposition 1.2. *If the interaction forces between particles depend only on the distance between the particles, then they are conservative.*

Proof. By the assumption of the proposition, the interaction forces are of of the form

$$\boldsymbol{F}_{12} = f\boldsymbol{e}_{12}\,, \qquad \boldsymbol{F}_{21} = f\boldsymbol{e}_{21}\,, \tag{44}$$

where f is a function of one variable (the distance between the particles) and

$$\boldsymbol{e}_{12} = \frac{\boldsymbol{r}_2 - \boldsymbol{r}_1}{|\boldsymbol{r}_2 - \boldsymbol{r}_1|}\,, \qquad \boldsymbol{e}_{21} = -\boldsymbol{e}_{12}\,. \tag{45}$$

We define the potential U of the system as

$$U(\boldsymbol{r}_1, \boldsymbol{r}_2) := u(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)\,, \qquad \text{where} \qquad u(s) := \int_{s_0}^{s} f(\rho)d\rho\,. \tag{46}$$

We then have by the chain rule

$$\begin{aligned}\frac{\partial U}{\partial x_1}(\boldsymbol{r}_1, \boldsymbol{r}_2) &= \frac{du}{ds}(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)\frac{\partial}{\partial x_1}|\boldsymbol{r}_1 - \boldsymbol{r}_2| \\ &= f(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)\frac{x_1 - x_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|}\,,\end{aligned} \tag{47}$$

where we used

$$\frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}\,. \tag{48}$$

It follows that

$$\begin{aligned}\nabla_1 U(\boldsymbol{r}_1, \boldsymbol{r}_2) &= f(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)\frac{\boldsymbol{r}_1 - \boldsymbol{r}_2}{|\boldsymbol{r}_1 - \boldsymbol{r}_2|} \\ &= -f(|\boldsymbol{r}_1 - \boldsymbol{r}_2|)\boldsymbol{e}_{12} \\ &= -\boldsymbol{F}_{12}(\boldsymbol{r}_1, \boldsymbol{r}_2)\,,\end{aligned} \tag{49}$$

where we used the notation

$$\nabla_1 = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial z_1} \end{pmatrix}. \tag{50}$$

Analogously we find

$$\nabla_2 U(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\boldsymbol{F}_{21}(\boldsymbol{r}_1, \boldsymbol{r}_2)\,, \tag{51}$$

i.e. the forces are conservative. □

Remark: (49), (51) can be written as

$$\nabla U = -\boldsymbol{F}\,, \tag{52}$$

where

$$\nabla = \begin{pmatrix} \nabla_1 \\ \nabla_2 \end{pmatrix}, \qquad \boldsymbol{F} = \begin{pmatrix} \boldsymbol{F}_{12} \\ \boldsymbol{F}_{21} \end{pmatrix}. \tag{53}$$

We have

Proposition 1.3. *If the interaction forces between particles depend only on the distance between them and in addition the external forces are conservative, i.e.*

$$\boldsymbol{F}_{ie} = -\nabla_i U_e\,, \tag{54}$$

then the total energy of the system

$$E = T + U + U_e \tag{55}$$

is conserved.

Proof. Using

$$\frac{d}{dt} m_i v_i^2 = 2 m_i \boldsymbol{v}_i \cdot \dot{\boldsymbol{v}}_i\,, \qquad \nabla_i U = -\boldsymbol{F}_{ij}\,, \qquad \nabla_i U_e = -\boldsymbol{F}_{ie}\,, \tag{56}$$

together with the equations of motion (26), (27), we obtain

$$\frac{dT}{dt} = \boldsymbol{v}_1 \cdot (m_1 \dot{\boldsymbol{v}}_1 - \boldsymbol{F}_{12} - \boldsymbol{F}_{1e}) + \boldsymbol{v}_2 \cdot (m_2 \dot{\boldsymbol{v}}_2 - \boldsymbol{F}_{21} - \boldsymbol{F}_{2e}) = 0\,. \tag{57}$$

□

Appendix A. Mathematical Identities

Product rule for functions of one variable $f(t)$, $g(t)$ and vector valued functions of one variable $\boldsymbol{a}(t)$, $\boldsymbol{b}(t)$:

$$\frac{d}{dt}(fg) = \frac{df}{dt}g + f\frac{dg}{dt}\,, \tag{58}$$

$$\frac{d}{dt}(\boldsymbol{a}\cdot\boldsymbol{b}) = \frac{d\boldsymbol{a}}{dt}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\frac{d\boldsymbol{b}}{dt}\,, \tag{59}$$

$$\frac{d}{dt}(\boldsymbol{a}\times\boldsymbol{b}) = \frac{d\boldsymbol{a}}{dt}\times\boldsymbol{b} + \boldsymbol{a}\times\frac{d\boldsymbol{b}}{dt}\,, \tag{60}$$

where we did not write the argument t.

Chain rule for functions of one variable $f(t)$, $g(t)$ and a function of several variables $F(x, y, z)$ composed with a vector valued function of one variable $\boldsymbol{r}(t)$:

$$\frac{d}{dt}f(g(t)) = f'(g(t))\frac{dg}{dt}(t)\,, \tag{61}$$

$$\frac{d}{dt}F(\boldsymbol{r}(t)) = \nabla F(\boldsymbol{r}(t))\cdot\frac{d\boldsymbol{r}}{dt}(t)\,, \tag{62}$$

where we denote by f' the derivative of f with respect to its argument.