LAGRANGIAN MECHANICS
PROBLEM SETS

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In the following we use boldface symbols to denote vectors.

1. Hand in:4.3.26, count: 15Pts.

1.1. **Rotating Reference Frame.** Let $\boldsymbol{r}(t)$ describe the two-dimensional motion of a particle in an inertial frame. Let $\boldsymbol{s}(t)$ describe the same motion but in a reference frame which is related to the inertial frame by $\boldsymbol{r} = R\boldsymbol{s}$, where

$$R = R(t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}.$$

I.e. $\boldsymbol{s}(t)$ describes the motion of the particle in a reference frame which rotates counter clockwise relative to the inertial frame with angular velocity ω.

(i) The equation of motion in the inertial frame is given by $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$. Show that the equation of motion in the rotating frame is

$$m\ddot{\boldsymbol{s}} = \boldsymbol{K} + \boldsymbol{F}_{cf} + \boldsymbol{F}_{co},$$

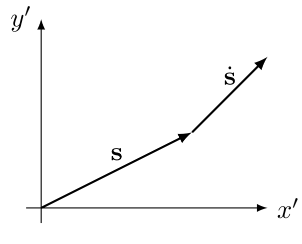
where $\boldsymbol{K} = R^{-1}\boldsymbol{F}$ is the acting force written in components of the rotating frame and

$$\boldsymbol{F}_{cf} = -mR^{-1}\ddot{R}\boldsymbol{s}, \qquad \boldsymbol{F}_{co} = -2mR^{-1}\dot{R}\dot{\boldsymbol{s}}$$

are the centrifugal and Coriolis forces respectively (both being inertial forces).

Hint: Take the time derivative of $\boldsymbol{r} = R\boldsymbol{s}$ twice and multiply the resulting equation with $R^{-1}(t) = R(-t)$ from the left.

(ii) Compute the inertial forces in terms of m, ω, $\boldsymbol{s}$ and $\dot{\boldsymbol{s}}$ and draw these two forces qualitatively in the following situation which shows the location and velocity of the particle in the rotating frame at a certain instant of time:



Hint: Compute the matrix products first, before multiplying with the vectors $\boldsymbol{s}$, $\dot{\boldsymbol{s}}$.

1.2. **Angular Momentum of System of Particles.** Show that the rate of change of the angular momentum of a system of particles equals the torque due to the external forces. I.e. show that

$$\dot{\boldsymbol{L}} = \boldsymbol{M},$$

where $\boldsymbol{L}$ and $\boldsymbol{M}$ are the angular momentum of the system of particles and the torque due to the external forces, respectively, given by

$$\boldsymbol{L} = \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{p}_i, \qquad \boldsymbol{M} = \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_{ie}.$$

Hint: Take the cross product of $\boldsymbol{r}_i$ with both sides of the equation of motion for the i'th particle:

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_{ie} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \boldsymbol{F}_{ij}$$

and sum over the index i. To convince yourself that the resulting double sum $\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \dots$ vanishes, consider first the case $n = 2$ and use Newtons third law.

1.3. **Decomposition of Kinetic Energy.** Show that the kinetic energy T of a system of n particles with masses m_i and velocities $v_i = |\dot{\boldsymbol{r}}_i|$

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$$

can be written as the sum of the kinetic energy of the center of mass motion T_R and the kinetic energy of the relative motion T_r, where

$$T_R = \frac{1}{2} M \dot{\boldsymbol{R}}^2, \qquad T_r = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\boldsymbol{r}}_{iR}^2.$$

Here M and $\boldsymbol{R}$ are the total mass and the center of mass of the system respectively. Also,

$$\boldsymbol{r}_{iR} = \boldsymbol{r}_i - \boldsymbol{R}$$

is the position of the i'th particle relative to the center of mass.

Hint: Compute the above expression for the kinetic energy T, using $\boldsymbol{r}_i = \boldsymbol{r}_{iR} + \boldsymbol{R}$ and the definition of the center of mass: $\boldsymbol{R} = \sum_i m_i \boldsymbol{r}_i / M$.