Vector Analysis

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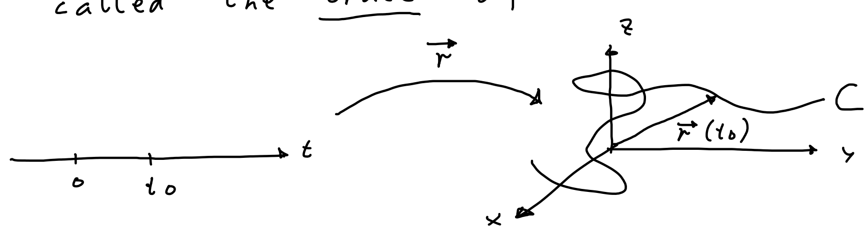
Parametric Curves in Space

Def.: A *curve* is a function:

$$\vec{r} : I \longrightarrow \mathbb{R}^3$$
$$t \longmapsto \vec{r}(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix},$$

where f, g, h are real valued functions and $I \subset \mathbb{R}$ is an Interval.

The set
$$C = \{ \vec{r}(t) : t \in I \}$$
is called the *trace* of the curve $\vec{r}(t)$.



Remarks:

(i) Arguments of $\vec{r}(t)$ in $\mathbb{R}$.

(ii) Function values in $\mathbb{R}^3$, i.e. curve is *vector valued function*.

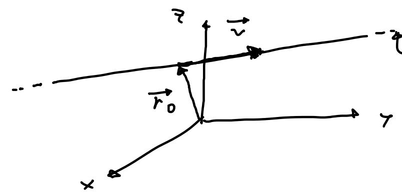
(iii) $\vec{r} : I \longrightarrow \mathbb{R}^2$ is curve in $(x\text{-}y\text{-})$ plane.

(iv) t is called *curve parameter*.

(v) Colloquially speaking: no distinction between curve and its trace.

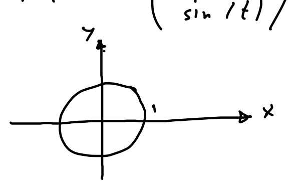
Examples:

(i) $\vec{r}(t) = \vec{r}_0 + \vec{v}\,t$; $I = \mathbb{R}$



trace C: straight line

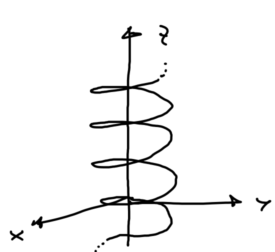
(ii) $\vec{r}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$; $I = [0, 2\pi)$.



unit circle at origin

(iii) $\vec{r}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ t \end{pmatrix}$

$I = \mathbb{R}$



helix/spiral

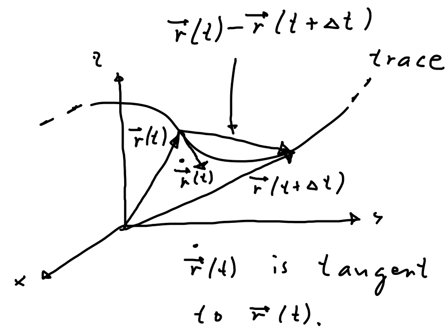
Physical interpretation

One possibility: Motion in space:

t : time

$\vec{r}(t)$: position of particle in space depending on time t.

Derivative



Def.:
$$\dot{\vec{r}}(t) = \lim_{\Delta t \to 0} \frac{\vec{r}(t+\Delta t) - \vec{r}(t)}{\Delta t}$$

By taking the limit component-wise, we obtain:
$$\dot{\vec{r}}(t) = \begin{pmatrix} \dot{f}(t) \\ \dot{g}(t) \\ \dot{h}(t) \end{pmatrix}$$
(i.e. derivative is taken component-wise.)

Physical interpretation

t : time

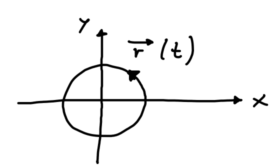
$\vec{r}(t)$: position

$\vec{v}(t) = \dot{\vec{r}}(t)$: velocity

$|\vec{v}(t)|$: speed

$\vec{a}(t) = \dot{\vec{v}}(t) = \ddot{\vec{r}}(t)$: acceleration

Example: Circular motion in plane with constant angular velocity ω, given by:



$$\vec{r}(t) = \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \end{pmatrix}$$

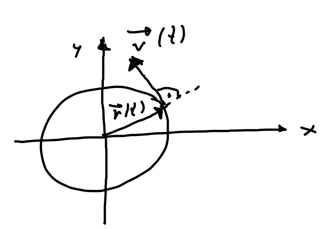
We confirm that the motion is on the circle by:
$$|\vec{r}(t)| = \sqrt{R^2\cos^2(\omega t) + R^2\sin^2(\omega t)} = R$$

The velocity is given by: $\vec{v}(t) = \begin{pmatrix} -R\omega\sin(\omega t) \\ R\omega\cos(\omega t) \end{pmatrix}$

We have: $\vec{r}(t) \cdot \vec{v}(t) = -R^2\cos(\ldots)\sin(\ldots) + R^2\sin(\ldots)\cos(\ldots)$
$= 0$,

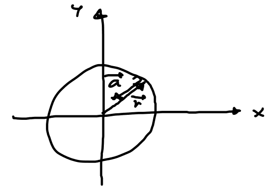
i.e. $\vec{r}(t) \perp \vec{v}(t)$

(perpendicular)



Furthermore:

$$\vec{a}(t) = \begin{pmatrix} -Rw^2 \cos(\dots) \\ -Rw^2 \sin(\dots) \end{pmatrix} = -w^2 \vec{r}(t)$$



Remark: $\vec{r} \perp \vec{v}$ can be shown, based only on $|\vec{r}(t)| = R$ with $R = \text{const.}$:

$$
\begin{aligned}
0 = \frac{d}{dt} R^2 = \frac{d}{dt} |\vec{r}(t)|^2 &= \frac{d}{dt}\left(\vec{r}(t) \cdot \vec{r}(t)\right) \\
&= \frac{d\vec{r}}{dt}(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \frac{d\vec{r}}{dt}(t) \\
&= \vec{v}(t) \cdot \vec{r}(t) + \vec{r}(t) \cdot \vec{v}(t) \\
&= 2\,\vec{r}(t) \cdot \vec{v}(t)
\end{aligned}
$$

and this computation is also valid in 3d (motion on sphere).

Integral

Def.: Let $\vec{r}(t) = \begin{pmatrix} f(t) \\ g(t) \\ h(t) \end{pmatrix}$, we define:

$$\int_a^b \vec{r}(t)\,dt = \begin{pmatrix} \int_a^b f(t)\,dt \\ \int_a^b g(t)\,dt \\ \int_a^b h(t)\,dt \end{pmatrix}$$

and similarly for the indefinite integral:

$$\int \vec{r}(t)\,dt = \vec{R}(t) + \vec{C},$$

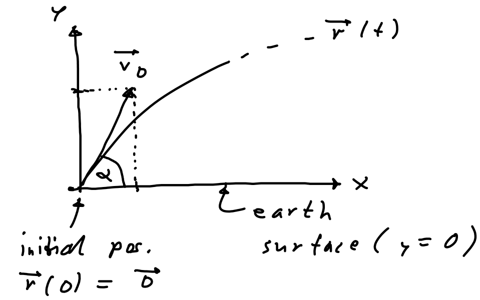
where $\vec{C}$ is a constant vector and $\dot{\vec{R}}(t) = \vec{r}(t)$ (i.e. $\vec{R}(t)$ is an antiderivative of $\vec{r}(t)$).

Examples:

$$\int \begin{pmatrix} \cos(t) \\ 1 \\ -2t \end{pmatrix} dt = \begin{pmatrix} \sin(t) \\ t \\ -t^2 \end{pmatrix} + \underbrace{\begin{pmatrix} C_1 \\ C_2 \\ C_3 \end{pmatrix}}_{=\vec{C}}$$

$$\int_0^{\pi} \begin{pmatrix} \cos(t) \\ 1 \\ -2t \end{pmatrix} dt = \begin{pmatrix} \sin(t) \\ t \\ -t^2 \end{pmatrix}\Bigg|_0^{\pi} = \begin{pmatrix} 0 \\ \pi \\ -\pi^2 \end{pmatrix}$$

Physical application : Projectile Motion



An object is fired at some angle α from an initial position. Under the influence of gravity (close to the earth's surface).

We choose coord. s.t. :

- motion in x-y-plane
- earth surface : $y = 0$
- initial pos. : $\vec{r}(0) = \vec{0}$

Initial velocity in this plane : $\vec{v}(0) = \vec{v}_0 = \begin{pmatrix} v_0 \cos(\alpha) \\ v_0 \sin(\alpha) \end{pmatrix}$,

where $v_0 = |\vec{v}_0|$ is the initial speed.

The physical situation is governed by Newton's law of motion:

$$\vec{F} = m\vec{a} , \quad \text{i.e.} \quad \underline{\vec{F} = m\ddot{\vec{r}}} , \quad \text{where} \quad \vec{F} = \begin{pmatrix} 0 \\ -mg \end{pmatrix}$$

mass ↑ ↑ grav. const.

$$\longrightarrow m\ddot{\vec{r}} = \begin{pmatrix} 0 \\ -mg \end{pmatrix} \longrightarrow \ddot{\vec{r}} = \begin{pmatrix} 0 \\ -g \end{pmatrix}$$

$$\longrightarrow \dot{\vec{r}}(t) = \int \ddot{\vec{r}}(t)\,dt = \begin{pmatrix} 0 \\ -gt \end{pmatrix} + \vec{v}_0$$

$$\longrightarrow \vec{r}(t) = \int \dot{\vec{r}}(t)\,dt = \begin{pmatrix} 0 \\ -\frac{1}{2}gt^2 \end{pmatrix} + \vec{v}_0 t + \vec{r}_0$$

I.e.

$$\underline{\vec{r}(t) = \begin{pmatrix} v_0 \cos(\alpha)\, t \\ v_0 \sin(\alpha)\, t - \frac{1}{2} g t^2 \end{pmatrix}} = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$$

From $x(t) = v_0 \cos(\alpha)\, t \longrightarrow t = \dfrac{x}{v_0 \cos(\alpha)}$.

This in $y(t)$:

$$\begin{aligned} y &= v_0 \sin(\alpha)\, t - \frac{1}{2} g t^2 \\ &= \frac{v_0 \sin(\alpha)}{v_0 \cos(\alpha)}\, x - \frac{1}{2} g \frac{1}{v_0^2 \cos^2(\alpha)} x^2 \\ &= \tan(\alpha)\, x - \frac{g}{2 v_0^2 \cos^2(\alpha)} x^2 , \end{aligned}$$

which is of the form: $y = ax^2 + bx + c$,

i.e. the trace of $\vec{r}(t)$ is a parabola in the x-y-plane.

Arc Length

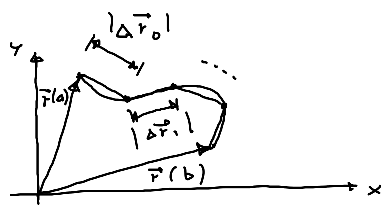
Let $\vec{r} : [a,b] \longrightarrow \mathbb{R}^3$
$\qquad t \longmapsto \vec{r}(t)$

Def.:

$$L = \int_a^b |\dot{\vec{r}}(t)|\, dt$$ is called the <u>arc length</u> of the curve $\vec{r}(t)$.

Interpretation

We give an interpretation of the above definition of arc length by computing the arc length of a curve approximately, using a polygon made up of n segments:



The arc length of the polygon is:

$$L_n = \sum_{i=0}^{n-1} |\Delta\vec{r}_i| = \sum_{i=0}^{n-1} \left|\vec{r}(t_{i+1}) - \vec{r}(t_i)\right|$$

Taylor expansion: $\vec{r}(t_{i+1}) = \vec{r}(t_i) + \frac{d\vec{r}}{dt}(t_i)(t_{i+1}-t_i) + O((t_{i+1}-t_i)^2)$

$$= \sum_{i=0}^{n-1} \left| \frac{d\vec{r}}{dt}(t_i)(t_{i+1}-t_i) + O((t_{i+1}-t_i)^2) \right|$$

$$= \sum_{i=0}^{n-1} \left| \frac{d\vec{r}}{dt}(t_i)\,\Delta t_i + O(\Delta t_i^2) \right|$$

$$\overset{\Delta t_i = \frac{b-a}{n}}{=} \frac{b-a}{n} \sum_{i=0}^{n-1} \left| \frac{d\vec{r}}{dt}(t_i) + O(\Delta t_i) \right|$$

$$\leq \underbrace{\frac{b-a}{n} \sum_{i=0}^{n-1} \left|\frac{d\vec{r}}{dt}(t_i)\right|}_{\xrightarrow{n\to\infty} \int_a^b \left|\frac{d\vec{r}}{dt}(t)\right| dt} + \underbrace{\frac{b-a}{n} \underbrace{\sum_{i=0}^{n-1} |O(\Delta t_i)|}_{\leq C(b-a)}}_{\xrightarrow{n\to\infty} 0}$$

I.e. the limit of the arc length of the polygon coincides with our above definition.

Arc length parameter:

Def.: Let $\vec{r} : [a,b] \longrightarrow \mathbb{R}^3$ be a curve.
The arc length parameter is the function:

$$s(t) = \int_a^t |\dot{\vec{r}}(\tau)| \, d\tau$$

Remarks: (i) We have $s(a) = 0$
$s(b) = L$, i.e.

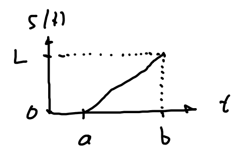
(ii) $s(t)$ is the arc length of the curve:

$$\vec{r} : [a,t] \longrightarrow \mathbb{R}^3$$
$$\tau \longmapsto \vec{r}(\tau)$$

(iii) Kinematically: $s(t)$ is the length of the trajectory travelled between $t=0$ and t (compare to $\vec{r}(t)$: position).

(iv) We have $\frac{ds}{dt}(t) = |\dot{\vec{r}}(t)| = |\vec{v}(t)|$: speed.

Reparametrization by arc length



Let us assume: $\dot{s}(t) = |\vec{v}(t)| > 0$.

$\longrightarrow$ $s(t)$ monotonically increasing:

$\longrightarrow$ $s(t)$ invertible, i.e. $t(s)$ exists.

$\longrightarrow$ We can use s as curve parameter:

$$\tilde{\vec{r}} : [0, L] \longrightarrow \mathbb{R}^3$$
$$s \longmapsto \tilde{\vec{r}}(s) = \vec{r}(t(s))$$

Remarks: (i) The traces of $\vec{r}$ and $\tilde{\vec{r}}$ are the same.

(ii) Any monotonically increasing function of t can be used for reparametrizing a curve (see also the invariance of the line integral under reparametrization below).

Example:

$$\vec{r}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ t \end{pmatrix} \; ; \quad I = [0, b]$$

$$s(t) = \int_0^t |\dot{\vec{r}}(\tau)| \, d\tau = \int_0^t \sqrt{2} \, d\tau = t\sqrt{2} \longrightarrow t(s) = \frac{s}{\sqrt{2}}$$

$$|\dot{\vec{r}}(\tau)| = \sqrt{\sin^2(\tau) + \cos^2(\tau) + 1} = \sqrt{2}$$

$$\longrightarrow \tilde{\vec{r}}(s) = \vec{r}(t(s)) = \begin{pmatrix} \cos\left(\frac{s}{\sqrt{2}}\right) \\ \sin\left(\frac{s}{\sqrt{2}}\right) \\ \frac{s}{\sqrt{2}} \end{pmatrix}$$

Line integrals of scalar and vector fields

Scalar fields

<u>Def.:</u> A <u>scalar field</u> ϕ is a function:

$$\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$$
$$(x,y,z) \longmapsto \phi(x,y,z)$$

<u>Examples:</u>
- Temperature in a room.
- Gravitational potential
  (i) close to earth surface: $\phi(x,y,z) = gz$
  (ii) in general: $\phi(x,y,z) = \dfrac{GM}{\sqrt{x^2+y^2+z^2}}$
- Electrical potential $\phi(x,y,z)$
- Density of a gas $\rho(x,y,z)$

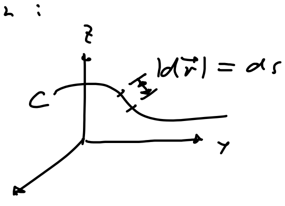
Line integral of scalar fields

<u>Def.:</u> Let $\vec{r} : [a,b] \longrightarrow \mathbb{R}^3$ be a curve with trace C and let $f : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a scalar field. Then, the <u>line integral of f over C</u> is:

$$\int_C f \, ds = \int_a^b f(\vec{r}(t)) \, |\dot{\vec{r}}(t)| \, dt$$

<u>Remark:</u> The appearance of $|\dot{\vec{r}}(t)|$ can be intuitively understood by the formal cancellation:

$$|\dot{\vec{r}}(t)| \, dt = \left| \frac{d\vec{r}}{\cancel{dt}}(t) \right| \cancel{dt} = |d\vec{r}| = ds$$



<u>Examples:</u>

(i) $f \equiv 1 \longrightarrow \int_C f \, ds = \int_a^b |\dot{\vec{r}}(t)| \, dt = L$ (arc length)

(ii) $f = \rho$, with ρ defined only along the curve and $\rho(x,y,z) =$ mass per unit length of the curve $\longrightarrow$

$$\int_C \underbrace{\rho \overbrace{ds}^{\text{Length element}}}_{= dm: \text{ Mass element}} = \int_a^b \rho(\vec{r}(t)) |\dot{\vec{r}}(t)| \, dt = \text{Mass}$$

Vector fields

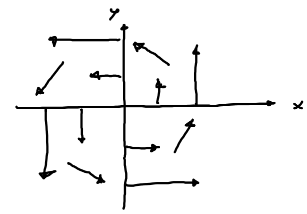
Def.: A vector field is a function:

$$\vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$$
$$(x,y,z) \longmapsto \vec{F}(x,y,z) = \begin{pmatrix} F_1(x,y,z) \\ F_2(x,y,z) \\ F_3(x,y,z) \end{pmatrix},$$

where F_1, F_2, F_3 are scalar fields.

Example: Velocity vector field of solid rotation around z-axis:

$$\vec{F} = \omega \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix}$$



We see: $|\vec{F}| = \omega\sqrt{x^2+y^2} = \omega r$

$\vec{F} \cdot \begin{pmatrix} x \\ y \\ 0 \end{pmatrix} = 0$, i.e. ω is the angular frequency and $\vec{F}$ points in circumferential direction.

Gradient (of scalar field)

Let $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a scalar field.

We define:
$$\vec{\nabla}\phi = \begin{pmatrix} \frac{\partial \phi}{\partial x} \\ \frac{\partial \phi}{\partial y} \\ \frac{\partial \phi}{\partial z} \end{pmatrix}$$

to be the gradient of ϕ (at (x,y,z))

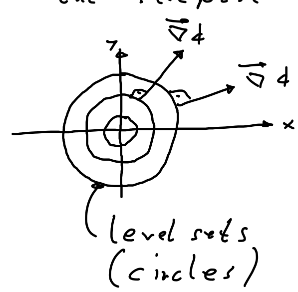
We recall the following properties of the gradient:

(i) $\vec{\nabla}\phi$ is perpendicular to the level sets of ϕ.

(ii) $\vec{\nabla}\phi$ points in the direction of the steepest increase of ϕ.

Example: $\phi(x,y) = x^2 + y^2$

$\longrightarrow \vec{\nabla}\phi = \begin{pmatrix} 2x \\ 2y \end{pmatrix}$



We see: $\vec{\nabla}\phi$ assigns to any point (x,y,z) the vector $\vec{\nabla}\phi(x,y,z)$ $\longrightarrow$ it is a vector field.

Def.: A vector field $\vec{F}$ is called a <u>gradient field</u> if there exists a scalar field $\phi : \mathbb{R}^3 \longrightarrow \mathbb{R}$, such that: $\vec{F} = \vec{\nabla}\phi$.

If $\vec{F}$ is a gradient field, then the corresponding ϕ is called a <u>potential function</u> for $\vec{F}$.

Example: $\vec{F} = \begin{pmatrix} y \\ x \end{pmatrix}$ is a gradient field with potential function $\phi(x,y) = xy$.

Line integrals of vector fields

Def.: Let $\vec{r} : [a,b] \longrightarrow \mathbb{R}^3$ be a curve with trace C and let $\vec{F} : \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a vector field. Then the <u>line integral of $\vec{F}$ over C</u> is:

$$\int_C \vec{F} \cdot \vec{T}\, ds ,$$

where $\vec{T}$ is the unit tangent vector of $\vec{r}(t)$, i.e.

$$\vec{T}(t) = \frac{\dot{\vec{r}}(t)}{|\dot{\vec{r}}(t)|} .$$

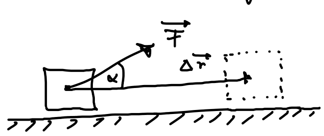
I.e. the line integral is given by:

$$\int_C \vec{F} \cdot \vec{T}\, ds \underset{\substack{\uparrow \\ \text{Def. of line} \\ \text{integral of} \\ \text{scalar field}}}{=} \int_a^b \vec{F}(\vec{r}(t)) \cdot \vec{T}(t) |\dot{\vec{r}}(t)|\, dt = \int_a^b \vec{F}(\vec{r}(t)) \cdot \dot{\vec{r}}(t)\, dt \quad \left(\begin{matrix}\text{use this} \\ \text{for computation}\end{matrix}\right)$$

$$= \int_C \vec{F} \cdot d\vec{r} \qquad \left(\begin{matrix}\text{good} \\ \text{notation}\end{matrix}\right)$$

Interpretation (in terms of mechanical work):

A body is displaced $\Delta\vec{r}$ with acting force $\vec{F}$:



The mechanical work done on the body by the force $\vec{F}$ is:

$$\Delta W = |\vec{F}| \cos(\alpha) |\Delta\vec{r}| = |\vec{F}||\Delta\vec{r}| \frac{\vec{F} \cdot \Delta\vec{r}}{|\vec{F}||\Delta\vec{r}|} = \vec{F} \cdot \Delta\vec{r}$$

Infinitesimally: $dW = \vec{F} \cdot d\vec{r}$

$$\longrightarrow W = \int_C \vec{F} \cdot d\vec{r}$$

Example: Work done by magnetic field on charged particle:

Lorentz force:

$\vec{F} = q\,\vec{v} \times \vec{B}$, where: q: Particle charge

$\vec{v}$: Particle velocity

$\vec{B}$: Magnetic field

$$\longrightarrow W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F} \cdot \underset{\vec{v}}{\underset{\|}{\dot{\vec{r}}}}\, dt = \int_a^b \underbrace{q(\vec{v} \times \vec{B}) \cdot \vec{v}}_{=0}\, dt = 0$$

I.e. the magnetic field does not do any work on the particle!

(Remark: $\vec{v}$ changes, but not $|\vec{v}|$

$\longrightarrow$ Kinetic energy $E_{kin} = \frac{m|\vec{v}|^2}{2}$ is constant)

Proposition: The line integrals of scalar and vector fields are invariant of reparametrization.

Proof:

Let $\vec{r} : [a,b] \longrightarrow \mathbb{R}^3 ; t \longmapsto \vec{r}(t)$ be a curve.

Let $u = u(t)$ be an increasing function of t

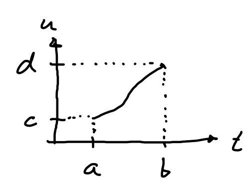
$\longrightarrow$ the inverse function $t(u)$ exists.

We define the reparametrized curve $\tilde{\vec{r}}(u)$ as follows:

$\tilde{\vec{r}} : [c,d] \longrightarrow \mathbb{R}^3$

$u \longmapsto \tilde{\vec{r}}(u) = \vec{r}(t(u))$, where $c = u(a)$

$d = u(b)$



$\longrightarrow \vec{r}, \tilde{\vec{r}}$ have the same trace: C

We have:

$$\int_C f\, ds = \underline{\int_a^b f(\vec{r}(t)) \left|\dot{\vec{r}}(t)\right| dt}$$

$$\underset{t=t(u)}{=} \int_{c=u(a)}^{d=u(b)} f(\vec{r}(t(u))) \underbrace{\left|\dot{\vec{r}}(t(u))\right| \frac{dt}{du}(u)}\, du = \underline{\int_c^d f(\tilde{\vec{r}}(u)) \left|\dot{\tilde{\vec{r}}}(u)\right| du}$$

$$\underset{\frac{dt}{du} > 0}{=} \left|\dot{\vec{r}}(t(u)) \frac{dt}{du}(u)\right| = \left|\frac{d}{du} \vec{r}(t(u))\right| = \left|\dot{\tilde{\vec{r}}}(u)\right|$$

The underlined expressions being therefore equal shows the independence. The same argument holds for line integrals of vector fields. $\square$

<u>Physical interpretation:</u>

If $\vec{r}(t)$ describes a motion through space, where a force field $\vec{F}$ is acting on the body, then the mechanical work done by the force field on the body does <u>not</u> depend how (with respect to time) the curve is traced out.

Example:
$$\vec{r} : [a,b] \longrightarrow \mathbb{R}^3$$
$$t \longmapsto \vec{r}(t)$$
$$\tilde{\vec{r}} : [\tfrac{a}{2}, \tfrac{b}{2}] \longrightarrow \mathbb{R}^3$$
$$t \longmapsto \tilde{\vec{r}}(t) = \vec{r}(2t)$$

$\vec{r}, \tilde{\vec{r}}$ have the same trace, but $\tilde{\vec{r}}(t)$ is <u>twice as fast</u>!

$$W = \int_C \vec{F} \cdot \vec{T}\, ds \quad \text{is the same for } \vec{r} \;\&\; \tilde{\vec{r}}.$$

<u>Remark:</u> The independence on parametrization is the reason why in the definitions

$$\int_C f\, ds \quad ; \quad \int_C \vec{F} \cdot \vec{T}\, ds$$

the actual curve $(\vec{r}, \tilde{\vec{r}}, \dots)$ does not show up!

<u>Dependence on orientation:</u>

The reparametrization in the above proposition considers only changes in parameters by monotonically increasing functions. In contrast to this, let us consider the two curves:

$$\vec{r} : [0,\ell] \longrightarrow \mathbb{R}^3 \qquad\qquad \tilde{\vec{r}} : [0,\ell] \longrightarrow \mathbb{R}^3$$
$$t \longmapsto \vec{r}(t) \qquad\qquad u \longmapsto \tilde{\vec{r}}(u) = \vec{r}(\ell - u)$$

These two curves possess the same trace but while the initial point of $\vec{r}$ corresponds to the final point of $\tilde{\vec{r}}$ and vice versa: $\vec{r}(0) = \tilde{\vec{r}}(\ell)$; $\vec{r}(\ell) = \tilde{\vec{r}}(0)$.

We say that these curves are of opposite <u>orientation</u>.

We investigate the behavior of line integras under the change of orientation:

Scalar fields:

$$\left|\dot{\tilde{\vec r}}(u)\right| = \left|\frac{d}{du}\tilde{\vec r}(u)\right| = \left|\frac{d}{du}\vec r(\ell-u)\right| = \left|\dot{\vec r}(\ell-u)\cdot(-1)\right| = \left|\dot{\vec r}(\ell-u)\right|$$

We have:

$$\int_{C^-} \phi\, ds = \int_0^\ell \phi(\tilde{\vec r}(u))\left|\dot{\tilde{\vec r}}(u)\right| du = \int_0^\ell \phi(\vec r(\ell-u))\left|\dot{\vec r}(\ell-u)\right| du$$

C^-: (Notation for opposite orientation w.r.t. C)

$t = \ell - u$, $\frac{dt}{du} = -1$

$$= -\int_\ell^0 \phi(\vec r(t))\left|\dot{\vec r}(t)\right| dt$$

$$= \int_0^\ell \phi(\vec r(t))\left|\dot{\vec r}(t)\right| dt = \int_C \phi\, ds$$

Vector fields:

$\dot{\tilde{\vec r}}(u) = -\dot{\vec r}(\ell-u)$

$$\int_{C^-} \vec F\cdot d\vec r = \int_0^\ell \vec F(\tilde{\vec r}(u))\cdot\dot{\tilde{\vec r}}(u)\, du = -\int_0^\ell \vec F(\vec r(\ell-u))\cdot\dot{\vec r}(\ell-u)\, du$$

$$= \int_\ell^0 \vec F(\vec r(t))\cdot\dot{\vec r}(t)\, dt$$

$$= -\int_0^\ell \vec F(\vec r(t))\cdot\dot{\vec r}(t)\, dt = \int_C \vec F\cdot d\vec r$$

I.e. we found:

$$\boxed{\int_C \phi\, ds = \int_{C^-} \phi\, ds \quad ; \quad \int_C \vec F\cdot d\vec r = -\int_{C^-} \vec F\cdot d\vec r}$$

Fundamental Theorem of Line Integrals

<u>Definition</u>: A set $\Omega \subset \mathbb{R}^2$ is called a <u>domain</u>, if:

(i) it is <u>open</u>: At every point in Ω we can draw a circle containing only points in Ω.

(ii) and <u>connected</u>: Between every pair of points in Ω there is a curve lying completely in Ω.

Theorem: Let $\vec{F}$ be a vector field defined in a domain Ω.
The following statements are equivalent:

(i) $\vec{F}$ is a gradient field.

(ii) $\int_C \vec{F}\cdot d\vec{r}$ is path independent.

(iii) $\int_C \vec{F}\cdot d\vec{r} = 0$ for any closed curve.

Proof <u>(i) ⇒ (ii)</u>

By assumption: $\vec{F} = \vec{\nabla}\phi$

Let now $\vec{r}:[a,b] \longrightarrow \mathbb{R}^2$ be any curve connecting two given points A, B.

$$\longrightarrow \int_C \vec{F}\cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t))\cdot \frac{d\vec{r}}{dt}(t)\,dt$$

$$\underset{\vec{F}=\vec{\nabla}\phi;\ \vec{r}(t)=\begin{pmatrix} x(t)\\ y(t)\end{pmatrix}}{=} \int_a^b \begin{pmatrix} \frac{\partial \phi}{\partial x}(\vec{r}(t)) \\ \frac{\partial \phi}{\partial y}(\vec{r}(t))\end{pmatrix}\cdot\begin{pmatrix} \frac{dx}{dt}(t) \\ \frac{dy}{dt}(t)\end{pmatrix} dt$$

$$= \int_a^b \left(\frac{\partial \phi}{\partial x}(\vec{r}(t))\,\frac{dx}{dt}(t) + \frac{\partial \phi}{\partial y}(\vec{r}(t))\,\frac{dy}{dt}(t)\right) dt$$

$$= \int_a^b \frac{d}{dt}\,\phi(\vec{r}(t))\,dt = \phi(\vec{r}(b)) - \phi(\vec{r}(a))$$

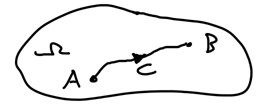
$$= \phi\big|_{\text{at } A} - \phi\big|_{\text{at } B}$$

The last expression depends only on the points A, B but not on the curve connecting them, this proves (i) ⇒ (ii).

<u>(ii) ⇒ (i)</u>

In order to show that from (ii) it follows that $\vec{F} = \vec{\nabla}\phi$, we have to construct the potential function ϕ. We pick (and fix) a point A in Ω and for any point B in Ω with coordinates (x,y) we set:

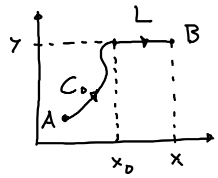
$$\phi(x,y) = \int_C \vec{F}\cdot d\vec{r},$$



where C is an arbitrary curve connecting A, B.
Because Ω is connected and $\int_C \vec{F}\cdot d\vec{r}$ is path independent (by assumption (ii)), $\phi(x,y)$ is well defined.

What we now show is that indeed $\vec{\nabla}\phi = \vec{F}$.

For this we consider the following curve C which connects A to B:



I.e. C is made up of C_0 and L, where L is parallel to the x-axis (it is always possible to choose such a curve because Ω is a domain).

$$\longrightarrow \quad \phi(x,y) = \int_C \vec{F}\cdot d\vec{r} = \underbrace{\int_{C_0} \vec{F}\cdot d\vec{r}}_{\text{does not depend on } x\,!} + \int_L \vec{F}\cdot d\vec{r}$$

L can be given by:

$$\vec{r} : [x_0, x] \longrightarrow \mathbb{R}^2$$
$$t \longmapsto \vec{r}(t) = \begin{pmatrix} t \\ y \end{pmatrix}$$

It follows that: $\dot{\vec{r}}(t) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\displaystyle\int_L \vec{F}\cdot d\vec{r} = \int_{x_0}^{x} \begin{pmatrix} F_1(t,y) \\ F_2(t,y) \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} dt$

$$= \int_{x_0}^{x} F_1(t,y)\, dt$$

I.e. $\displaystyle \phi(x,y) = \int_{C_0} \vec{F}\cdot d\vec{r} + \int_{x_0}^{x} F_1(t,y)\, dt$

$$\longrightarrow \quad \frac{\partial \phi}{\partial x}(x,y) = F_1(x,y)$$

Similarly, but choosing a curve C which contains a part parallel to the y-axis, we obtain:

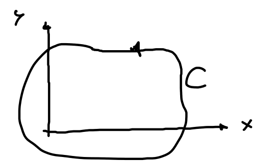
$$\frac{\partial \phi}{\partial y}(x,y) = F_2(x,y)$$

I.e. we have shown: $\quad \vec{\nabla}\phi = \vec{F}$.

(ii) $\Leftrightarrow$ (iii) is left as an exercise. $\square$

Green's Theorem

Definition: Let $\vec{r}(t)$ be a closed, non self intersecting curve in $\mathbb{R}^2$. We call $\vec{r}(t)$ <u>positively oriented</u> if $\vec{r}(t)$ moves in a counterclockwise direction (as t increases):

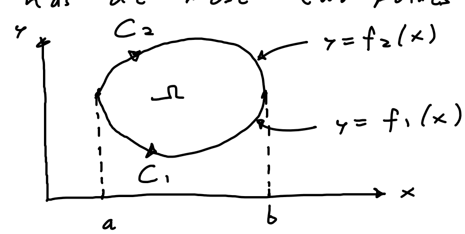


Theorem (Green): Let C be a positively oriented curve in $\mathbb{R}^2$ enclosing a region Ω. Let $\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix}$ be a vector field. Then:

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA}$$

Proof: The proof is in three stages of increasing complexity of the shape of Ω.

(i) The region Ω is of type: Any line parallel the x- or y-axis has at most two points in common with C:



The boundary of such a region can be given in terms of two graphs of functions:

$$y = f_1(x) \quad ; \quad y = f_2(x)$$

Corresponding curves: $\vec{r}_1(x) = \begin{pmatrix} x \\ f_1(x) \end{pmatrix}$; $x \in [a, b]$

$$\vec{r}_2(x) = \begin{pmatrix} x \\ f_2(x) \end{pmatrix} \ ; x \in [a, b]$$

$$\longrightarrow \int_C \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} + \int_{C_2^-} \vec{F} \cdot d\vec{r} = \int_{C_1} \vec{F} \cdot d\vec{r} - \int_{C_2} \vec{F} \cdot d\vec{r} \ ,$$

where we made use of the orientation property of the line integral of vector fields: $\int_C \vec{F}\cdot d\vec{r} = -\int_{C^-} \vec{F}\cdot d\vec{r}$.

Let now in a first step: $\vec{F}_1 = \begin{pmatrix} M \\ 0 \end{pmatrix}$

$$\longrightarrow \int_C \vec{F}_1 \cdot d\vec{r} = \int_{C_1} \vec{F}_1 \cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r}$$

$$= \int_a^b \vec{F}_1(\vec{r}_1(x)) \cdot \frac{d\vec{r}_1}{dt}(x)\,dx - \int_a^b \vec{F}_1(\vec{r}_2(t)) \cdot \frac{d\vec{r}_2}{dt}(x)\,dx$$

$$\left.\begin{matrix} \vec{F}_1 = \begin{pmatrix} M \\ 0 \end{pmatrix} \\ \vec{r}_1(x) = \begin{pmatrix} x \\ f_1(x) \end{pmatrix} \end{matrix}\right\} \curvearrowright \; = \int_a^b M(x, f_1(x))\,dx - \int_a^b M(x, f_2(x))\,dx$$

$$= \int_a^b \Big(\underbrace{M(x, f_1(x)) - M(x, f_2(x))}_{= \int_{f_2(x)}^{f_1(x)} \frac{\partial M}{\partial y}(x,y)\,dy} \Big)\,dx$$

$$= \int_a^b \int_{f_2(x)}^{f_1(x)} \frac{\partial M}{\partial y}(x,y)\,dy\,dx$$

$$= -\int_a^b \int_{f_1(x)}^{f_2(x)} \frac{\partial M}{\partial y}(x,y)\,dy\,dx = \iint_\Omega -\frac{\partial M}{\partial y}(x,y)\,dA$$

I.e., we found:

if $\vec{F}_1 = \begin{pmatrix} M \\ 0 \end{pmatrix} \longrightarrow \int_C \vec{F}_1 \cdot d\vec{r} = \iint_\Omega -\frac{\partial M}{\partial y}\,dA$

it follows similarly (but describing the boundary of Ω with two graphs of functions of

$x = g_1(y)$; $x = g_2(y)$) that:

if $\vec{F}_2 = \begin{pmatrix} 0 \\ N \end{pmatrix} \longrightarrow \int_C \vec{F}_2 \cdot d\vec{r} = \iint_\Omega \frac{\partial N}{\partial x}\,dA$

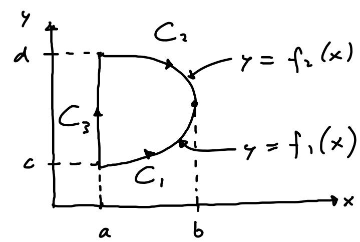
For a general $\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix}$, we have $\vec{F} = \vec{F}_1 + \vec{F}_2$

$$\longrightarrow \int_C \vec{F}\cdot d\vec{r} = \int_C (\vec{F}_1 + \vec{F}_2)\cdot d\vec{r} = \int_C \vec{F}_1\cdot d\vec{r} + \int_C \vec{F}_2 \cdot d\vec{r}$$

$$= \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA.$$

This proves the thm. for the first type of region Ω.

(ii) The region Ω is of type:



I.e. part of the boundary of Ω is parallel to one of the axes.

We have:

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} - \int_{C_2} \vec{F}\cdot d\vec{r} - \int_{C_3} \vec{F}\cdot d\vec{r} ,$$

where $\vec{r}_1, \vec{r}_2$ are given as in (i) and

$$\vec{r}_3(t) = \begin{pmatrix} a \\ t \end{pmatrix} \quad ; \quad t \in [c, d]$$

Hence $\dot{\vec{r}}_3(t) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$ and, restricting again to the case $\vec{F}_1 = \begin{pmatrix} M \\ 0 \end{pmatrix}$, we have:

$$\int_{C_3} \vec{F}_1\cdot d\vec{r} = \int_c^d \begin{pmatrix} M(\vec{r}_3(t)) \\ 0 \end{pmatrix}\cdot\begin{pmatrix} 0 \\ 1 \end{pmatrix} dt = 0 ,$$

which implies: $\displaystyle \int_C \vec{F}_1\cdot d\vec{r} = \iint_\Omega -\frac{\partial M}{\partial y}\, dA .$

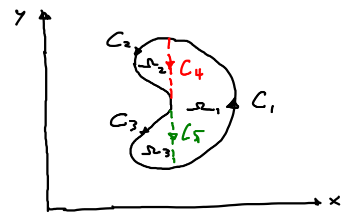
Using the reasoning as in (i), for $\vec{F}_2 = \begin{pmatrix} 0 \\ N \end{pmatrix}$ we obtain:

$$\int_C \vec{F}_2\cdot d\vec{r} = \iint_\Omega \frac{\partial N}{\partial x}\, dA$$

So if $\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix} = \vec{F}_1 + \vec{F}_2$, then:

$$\int_C \vec{F}\cdot d\vec{r} = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA .$$

(iii) Ω is of type:



The idea is to split Ω into $\Omega_1, \Omega_2, \Omega_3$, each of which is of type (ii).

Then:

$$\int_C \vec{F}\cdot d\vec{r} = \int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_2} \vec{F}\cdot d\vec{r} + \int_{C_3} \vec{F}\cdot d\vec{r}$$

$$= \underbrace{\int_{C_1} \vec{F}\cdot d\vec{r} + \int_{C_4} \vec{F}\cdot d\vec{r} + \int_{C_5} \vec{F}\cdot d\vec{r}} + \underbrace{\int_{C_2} \vec{F}\cdot d\vec{r} - \int_{C_4} \vec{F}\cdot d\vec{r}} + \underbrace{\int_{C_3} \vec{F}\cdot d\vec{r} - \int_{C_5} \vec{F}\cdot d\vec{r}}$$

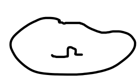
$$= \int_{\partial\Omega_1} \vec{F}\cdot d\vec{r} + \int_{\partial\Omega_2} \vec{F}\cdot d\vec{r} + \int_{\partial\Omega_3} \vec{F}\cdot d\vec{r}$$

$$= \iint_{\Omega_1} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA + \iint_{\Omega_2} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA + \iint_{\Omega_3} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA$$

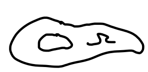
$$= \iint_{\Omega} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dA \qquad \square$$

Application to gradient fields

Definition: A domain Ω is called simply connected, if every closed curve in Ω encloses only points in Ω:



simply connected



not simply connected

Theorem: Let Ω be a simply connected domain. Let $\vec{F} = \binom{M}{N}$ be a vectorfield defined in Ω. Then, the following are equivalent:

(i) $\vec{F}$ is a gradient field,

(ii) $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0$ in Ω.

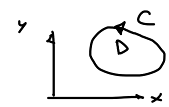
Proof: (i) $\Rightarrow$ (ii)

$$\vec{F} = \vec{\nabla}\phi \longrightarrow M = \frac{\partial \phi}{\partial x} \; ; \; N = \frac{\partial \phi}{\partial y}$$

$$\longrightarrow \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = \frac{\partial^2 \phi}{\partial x \partial y} - \frac{\partial^2 \phi}{\partial y \partial x} = 0 .$$

(ii) $\Rightarrow$ (i)

$$\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 0 \quad \text{in } \Omega$$



$\longrightarrow$ For any closed loop C:

$$\int_C \vec{F} \cdot d\vec{r} \underset{\uparrow \text{ Green}}{=} \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA = 0$$

Therefore, by the fundamental theorem of line integrals $\longrightarrow$ there exists ϕ such that $\vec{F} = \vec{\nabla}\phi$

$\square$

Example: Let: $\Omega = \mathbb{R}^2$; $\vec{F} = \begin{pmatrix} 1 + 4x - y \\ e^y - 2y - x \end{pmatrix} = \begin{pmatrix} M \\ N \end{pmatrix}$

We have: $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} = -1 - (-1) = 0$

$\longrightarrow$ There exists ϕ, such that $\vec{F} = \vec{\nabla}\phi$.

How to find ϕ:

$\vec{F} = \vec{\nabla}\phi$ with the above vector field is:

$$1 + 4x - y = \frac{\partial \phi}{\partial x} \qquad (1)$$

$$e^y - 2y - x = \frac{\partial \phi}{\partial y} \qquad (2)$$

$$(1) \longrightarrow \phi(x,y) = x + 2x^2 - xy + f(y)$$

$$\longrightarrow \frac{\partial \phi}{\partial y}(x,y) = -x + f'(y) \overset{(2)}{=} e^y - 2y - x$$

$$\longrightarrow f'(y) = e^y - 2y$$

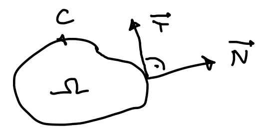
$$\longrightarrow f(y) = e^y - y^2 + \underbrace{C}_{\text{constant}}$$

$$\longrightarrow \underline{\phi(x,y) = x + 2x^2 - xy + e^y - y^2 + C}$$

Alternative form of Greens theorem

Derivation: Let $\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix}$; $\vec{G} = \begin{pmatrix} -N \\ M \end{pmatrix}$

In the following we denote by $\vec{T} = \begin{pmatrix} T_1 \\ T_2 \end{pmatrix}$ the unit tangent vector to C and by $\vec{N} = \begin{pmatrix} N_1 \\ N_2 \end{pmatrix}$ the unit outward pointing normal vector to C:



Then: $$\iint_\Omega \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA \underset{\text{Green with } \vec{G}}{=} \int_C \vec{G} \cdot d\vec{r}$$

$$= \int_C \vec{G} \cdot \vec{T} \, ds$$

$$= \int_C \begin{pmatrix} -N \\ M \end{pmatrix} \cdot \begin{pmatrix} T_1 \\ T_2 \end{pmatrix} ds$$

$$= \int_C \left(-N T_1 + M T_2 \right) ds$$

$\vec{N} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \vec{T} = \begin{pmatrix} T_2 \\ -T_1 \end{pmatrix}$ $$= \int_C \left(M N_1 + N N_2 \right) ds$$

$$= \int_C \vec{F} \cdot \underbrace{\vec{N}\, ds}_{= d\vec{N} \text{ (Notation)}} = \int_C \vec{F} \cdot d\vec{N}$$

I.e. we find:

$$\boxed{\int_C \vec{F} \cdot d\vec{N} = \iint_\Omega \left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right) dA}$$

Remarks: (i) This form is called the normal form of Greens theorem, while

$$\int_C \vec{F} \cdot d\vec{r} = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

is called the tangential form.

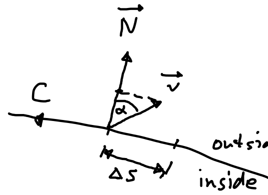
(ii) For any vector field $\vec{F} = \begin{pmatrix} F_1 \\ F_2 \end{pmatrix}$, the expression $\frac{\partial F_1}{\partial x} + \frac{\partial F_2}{\partial y}$ is called <u>divergence</u> of $\vec{F}$ and can formally be written using $\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$ as $\boxed{\vec{\nabla} \cdot \vec{F}}$

(iii) $\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{pmatrix}$ is called <u>nabla operator</u>.

(Recall the notation for the gradient: $\vec{\nabla}\phi$).

<u>Interpretation of $\int_C \vec{F} \cdot d\vec{N}$:</u>

Let $\vec{F} = \vec{v}$ be the velocity vector field of a two-dimensional, incompressible fluid flow (e.g. water in a shallow river).
We consider a small part of C in which we assume the velocity $\vec{v}$ being constant (This assumption will make sense once we consider an infinitesimal part of C):

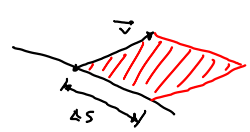


Then, the component of $\vec{v}$ which is normal to C is:

$$v_{\perp} = |\vec{v}| \cos(\alpha) = |\vec{v}| \frac{\vec{v} \cdot \vec{N}}{|\vec{v}||\vec{N}|} = \vec{v} \cdot \vec{N}.$$

($v_{\perp} > 0$ if $\vec{v}$ points outward, $v_{\perp} < 0$ if $\vec{v}$ points inward).
Then, the amount of fluid flowing out of Ω through the part of C under consideration in unit time is:

$$v_{\perp} \Delta s = \vec{v} \cdot \vec{N} \Delta s = ////$$



We call this the <u>outflux</u> through the part of C. The outflux throug an infinitesimally small part of C is then $\vec{v} \cdot \vec{N} \, ds$. Hence the <u>total outflux</u> out of Ω is:

$$\int_C \vec{v} \cdot \vec{N} \, ds = \int_C \vec{v} \cdot d\vec{N},$$

which is the left hand side of Green's thm. in normal form. Remark: $\int_C \vec{v} \cdot d\vec{N} < 0$ corresponds to influx.

Interpretation of divergence

Let $f(x,y) = \vec{\nabla} \cdot \vec{v}$

The Taylor expansion of $f(x,y)$ at $(x_0,y_0) \in \Omega$ is:

$$f(x,y) = f(x_0,y_0) + \left(\frac{\partial f}{\partial x}\right)_0 \Delta x + \left(\frac{\partial f}{\partial y}\right)_0 \Delta y + O\left((\Delta x^2), (\Delta y)^2, \Delta x \, \Delta y\right),$$

where we use the notation: $\Delta x = x - x_0$

$$\left(\frac{\partial f}{\partial x}\right)_0 = \frac{\partial f}{\partial x}(x_0,y_0).$$

Using now Greens theorem (normal form), together with the above interpretation of $\int_C \vec{v} \cdot d\vec{N}$ as outward flux, we obtain:

$$\frac{\text{outward flux}}{\text{area of } \Omega} = \frac{\int_C \vec{v} \cdot d\vec{N}}{\iint_\Omega dA} \overset{\text{Green}}{=} \frac{\iint_\Omega \vec{\nabla} \cdot \vec{v} \, dA}{\iint_\Omega dA} = \frac{\iint_\Omega f(x,y) \, dA}{\iint_\Omega dA}$$

$$= \frac{\iint \left(f(x_0,y_0) + \left(\frac{\partial f}{\partial x}\right)_0 \Delta x + \dots\right) dA}{\iint_\Omega dA}$$

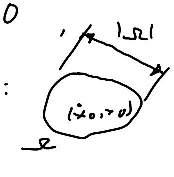
$$= f(x_0,y_0) + \frac{\left(\frac{\partial f}{\partial x}\right)_0 \iint_\Omega \Delta x \, dA}{\iint_\Omega dA} + \dots$$

Now, since:

$$\left| \frac{\left(\frac{\partial f}{\partial x}\right)_0 \iint_\Omega \Delta x \, dA}{\iint_\Omega dA} \right| \leq \frac{\left|\left(\frac{\partial f}{\partial x}\right)_0\right| |\Omega| \iint_\Omega dA}{\iint_\Omega dA}$$

$$= \left|\left(\frac{\partial f}{\partial x}\right)_0\right| |\Omega| \xrightarrow{\text{as } |\Omega| \to 0} 0$$

where $|\Omega|$ is the (one-dimensional) size of Ω:



and the limit $|\Omega| \to 0$ is understood such that $(x_0,y_0) \in \Omega$ while taking this limit.

Putting things together, we obtain:

$$\vec{\nabla}\cdot\vec{v}(x_0,y_0) = f(x_0,y_0) = \lim_{|\Omega|\to 0} \frac{\text{outward flux}}{\text{area of } \Omega}$$

I.e. $\vec{\nabla}\cdot\vec{v}$ can be interpreted as a <u>source density</u> (flux per unit area).

The over all interpretation of the normal form of Greens theorem can therefore be given as:

$$\boxed{\underbrace{\int_C \vec{v}\cdot d\vec{N}}_{\substack{\text{total}\\ \text{outward}\\ \text{flux}}} = \underbrace{\iint_\Omega \vec{\nabla}\cdot\vec{v}\, dA}_{\substack{\text{total}\\ \text{source}\\ \text{strength}}}}$$

Applications:

(i) <u>Gausse's Law in 2D</u>

Gauss's law (one of Maxwell's equations) is:

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$

where $\vec{E}$ is the electric field, ρ the charge density and ε_0 the permittivity of vacuum. Let Q be the total charge in some region $\Omega\subset\mathbb{R}^2$, i.e.:

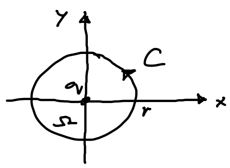
$$Q = \iint_\Omega \rho\, dA .$$

Then: $$\frac{Q}{\varepsilon_0} = \iint_\Omega \frac{\rho}{\varepsilon_0}\, dA \underset{\text{Gauss}}{=} \iint_\Omega \vec{\nabla}\cdot\vec{E}\, dA \underset{\text{Green}}{=} \int_C \vec{E}\cdot d\vec{N} .$$

I.e. the outflux of the electric field $\vec{E}$ out of a region Ω is equal to the total charge in Ω.

We use this to derive Coulombs law in two dimensions.

To do this we consider a point charge q sitting at the origin and a circle C with radius r and center at the origin:



$$\vec{r}(t) = \begin{pmatrix} r\cos(t) \\ r\sin(t) \end{pmatrix} \; ; \; t \in [0, 2\pi].$$

At any point $\vec{r}(t)$, the unit normal and tangential vectors are:

$$\vec{T} = \begin{pmatrix} -\sin(t) \\ \cos(t) \end{pmatrix} \; ; \; \vec{N} = \begin{pmatrix} \cos(t) \\ \sin(t) \end{pmatrix}$$

We have:

$$\frac{q}{\varepsilon_0} = \iint_{\Omega} \frac{\rho}{\varepsilon_0}\, dA \underset{\text{Gauss}}{\overset{\uparrow}{=}} \iint_{\Omega} \vec{\nabla}\cdot\vec{E}\, dA \underset{\text{Green}}{\overset{\uparrow}{=}} \int_C \vec{E}\cdot d\vec{N} = \int_C \vec{E}\cdot\vec{N}\, ds$$

Now, because of the symmetry of the physical problem, we assume:

(i) $\vec{E} \parallel \vec{N}$

(ii) $|\vec{E}|$ independent on the position on the circle.

Therefore:

$$\int_C \vec{E}\cdot\vec{N}\, ds \underset{(i)}{\overset{\uparrow}{=}} \int_C |\vec{E}|\underbrace{|\vec{N}|}_{=1}\, ds = \int_C |\vec{E}|\, ds = |\vec{E}|\int_C ds = |\vec{E}|\, 2\pi r$$

Together with $\frac{q}{\varepsilon_0} = \int_C \vec{E}\cdot\vec{N}\, ds$ we obtain:

$$|\vec{E}| = \frac{q}{2\pi r \varepsilon_0}$$

I.e., using (i): $\quad \vec{E}(x,y) = \dfrac{q}{2\pi\varepsilon_0(x^2+y^2)}\begin{pmatrix} x \\ y \end{pmatrix}$

The force acting on another charge q_2 is then:

$$\vec{F} = \vec{E} q_2 = \frac{q q_2}{2\pi \varepsilon_0 (x^2 + y^2)} \begin{pmatrix} x \\ y \end{pmatrix}$$

I.e. in absolute value:

$$|\vec{F}| = \frac{q q_2}{2\pi \varepsilon_0 \sqrt{x^2 + y^2}} = \frac{q q_2}{2\pi \varepsilon_0 r}$$

Remark: The $\frac{1}{r}$ dependency is directly related to the fact that we study a two-dimensional problem. We are going to revisit this problem in three dimensions and find there the well known $\frac{1}{r^2}$ dependency from the classical Coulomb's law.

(ii) <u>Gauss's law for magnetism</u>

Another one of Maxwells equations is:

$$\boxed{\vec{\nabla} \cdot \vec{B} = 0,}$$

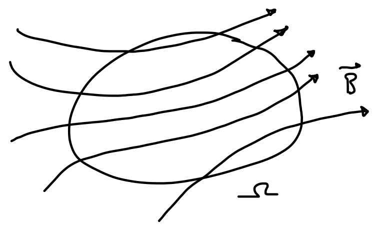
where $\vec{B}$ is the magnetic field (more precise: the magnetic flux density).

Using Green's theorem, we obtain from this:

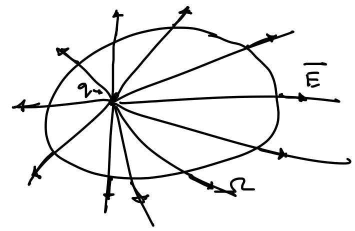
$$\int_C \vec{B} \cdot d\vec{N} = 0,$$

i.e. the total magnetic flux out of a region <u>is</u> always zero. Or put differently: There don't exist any sources for the magnetic field. This is in contrast to the electric field $\vec{E}$, where the sources are the charges.

Magnetic field lines:



Electric field lines:



$$\int_C \vec{B} \cdot d\vec{N} = 0 \qquad\qquad \int_C \vec{E} \cdot d\vec{N} = \frac{q}{\varepsilon_0}$$

(iii) <u>Continuity Equation</u>

We consider the flow of a compressible fluid described by the density ρ (a scalar field) and the fluid velocity $\vec{v}$ (a vector field), both depending on space (x,y) and time t.

The fluid mass in a region Ω at time t is:

$$M(t) = \int_\Omega \rho(t,x,y)\, dA$$

Therefore:

$$\text{rate of change of mass in } \Omega = \frac{dM}{dt}(t) = \iint_\Omega \frac{\partial \rho}{\partial t}(t,x,y)\, dA$$

$$\text{outflux through } \underbrace{\partial\Omega}_{\text{boundary of } \Omega} = \int_{\partial\Omega} (\rho\vec{v}) \cdot d\vec{N} \underset{\text{Green}}{=} \iint_\Omega \big(\vec{\nabla} \cdot (\rho\vec{v})\big)(t,x,y)\, dA$$

Conservation of mass requires:

$$\text{rate of change of mass in } \Omega = - \text{outflux through } \partial\Omega$$

I.e. $$\iint_\Omega \frac{\partial \rho}{\partial t}(t,x,y)\, dA = -\iint_\Omega \big(\vec{\nabla} \cdot (\rho\vec{v})\big)(t,x,y)\, dA$$

I.e. $$\iint_{\Omega} \left(\frac{\partial \rho}{\partial t}(t,x,y) + (\vec{\nabla} \cdot (\rho \vec{v}))(t,x,y) \right) dA = 0$$

Since this has to hold for arbitrary Ω, we conclude:

$$\frac{\partial \rho}{\partial t}(t,x,y) + (\vec{\nabla} \cdot (\rho \vec{v}))(t,x,y) = 0$$

i.e. (leaving away the arguments):

$$\boxed{\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{v}) = 0}$$ <u>Continuity equation</u>

Interpretation of the tangential form of Greens theorem

We recall the tangential form:

$$\int_C \vec{F} \cdot d\vec{r} = \iint_{\Omega} \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA$$

For the interpretation we again think of the vector field $\vec{F}$ being the velocity vector field of a two-dimensional incompressible fluid flow.

We define the <u>circulation</u> of $\vec{F}$ around C as: $\int_C \vec{F} \cdot d\vec{r}$.

The interpretation of this quantity is that it sums up the tangential component of $\vec{F}$ <u>around</u> C.

Thus:

$\int_C \vec{F} \cdot d\vec{r} > 0$: net counter clockwise motion

$\int_C \vec{F} \cdot d\vec{r} < 0$: net clockwise motion

Definition: The quantity $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$ is called the <u>curl density</u> of the vectorfield $\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix}$.

The interpretation of the curl density is now given, using the tangential form of Green's theorem: (this is analogous to the interpretation of the divergence $\vec{\nabla} \cdot \vec{F}$ using the normal form of Green's theorem).

$$\left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right)(x_0, y_0) = \lim_{|\Omega| \to 0} \frac{\text{circulation around } \Omega}{\text{Area of } \Omega}$$

I.e. the curl density is the <u>circulation density</u>: How much the fluid is rotating at this point.

<u>Remarks:</u> (i) We already gave an interpretation of $\int_C \vec{F} \cdot d\vec{r}$ in the case where $\vec{F}$ is a force field: mechanical work.

However, in this case there is no direct intuitive description what the curl density should be and for conservative force fields it vanishes anyway.

(ii) A deeper understanding of divergence and curl density of vector fields can be obtained by looking at typical examples of vector fields, see the respective problem set.

Surfaces

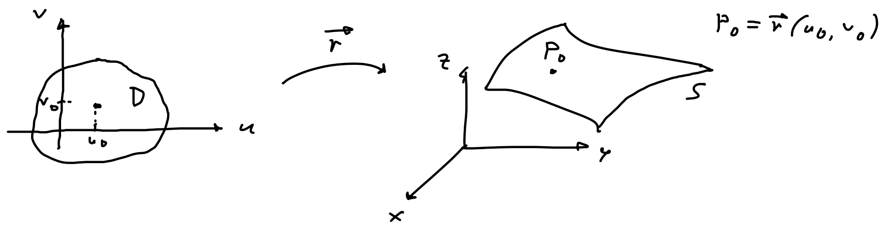
Def. A <u>parametric representation of a surface</u> is a function:

$$\vec{r} : D \longrightarrow \mathbb{R}^3$$
$$(u,v) \longmapsto \vec{r}(u,v) = \begin{pmatrix} f(u,v) \\ g(u,v) \\ h(u,v) \end{pmatrix},$$

where f, g, h are real valued functions, $D \subset \mathbb{R}^2$.

The set: $S = \{ \vec{r}(u,v) : (u,v) \in D \}$

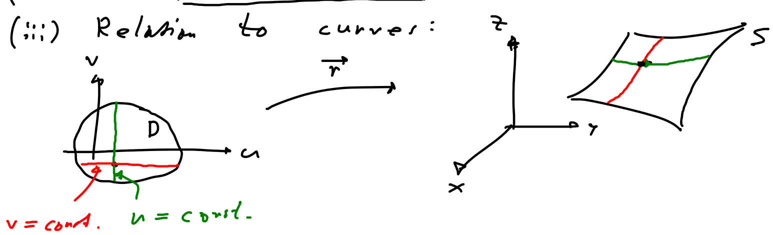
is called the <u>surface</u>.



Remarks: (i) u, v : <u>parameters</u>

(ii) D : <u>parameter domain</u>

(iii) Relation to curves:

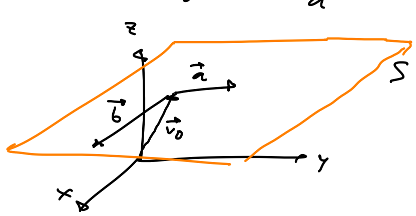


I.e. it is 2-D extension of curve.

Examples:

(i) $\vec{r} : \mathbb{R}^2 \longrightarrow \mathbb{R}^3$

$$(u,v) \longmapsto \vec{r}(u,v) = \underbrace{\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}}_{=\vec{v}_0} + u \underbrace{\begin{pmatrix} 2 \\ 1 \\ 0 \end{pmatrix}}_{\vec{a}} + v \underbrace{\begin{pmatrix} 17 \\ \sqrt{2} \\ -\pi \end{pmatrix}}_{\vec{b}}$$

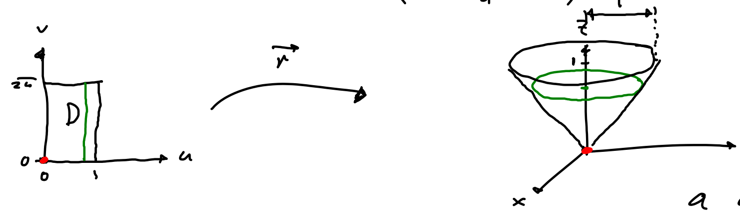


A <u>plane</u>

(In general: $\vec{r}(u,v) = \vec{v}_0 + u\vec{a} + v\vec{b}$)

(ii) $\quad \vec{r} : [0,1] \times [0, 2\pi] \longrightarrow \mathbb{R}^3$

$$(u,v) \longmapsto \vec{r}(u,v) = \begin{pmatrix} u\cos(v) \\ u\sin(v) \\ u \end{pmatrix}$$



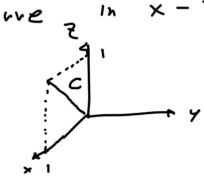
a cone!

Rem.: Usually we use r, φ as parameters: $\quad \vec{r}(r,\varphi) = \begin{pmatrix} r\cos(\varphi) \\ r\sin(\varphi) \\ r \end{pmatrix}$

(iii) Surfaces of revolution:

Take a curve in x-z-plane: $\quad \vec{r} : [0,1] \longrightarrow \mathbb{R}^3$

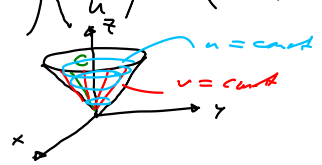
$$u \longmapsto \vec{r}(u) = \begin{pmatrix} u \\ 0 \\ u \end{pmatrix}$$



Then rotate around z-axis: $(v \in [0, 2\pi])$

$$\vec{r}(u,v) = \begin{pmatrix} \cos(v) & -\sin(v) & 0 \\ \sin(v) & \cos(v) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} u \\ 0 \\ u \end{pmatrix} = \begin{pmatrix} u\cos(v) \\ u\sin(v) \\ u \end{pmatrix}$$

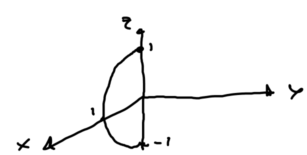
again our cone:



(iv) Sphere:

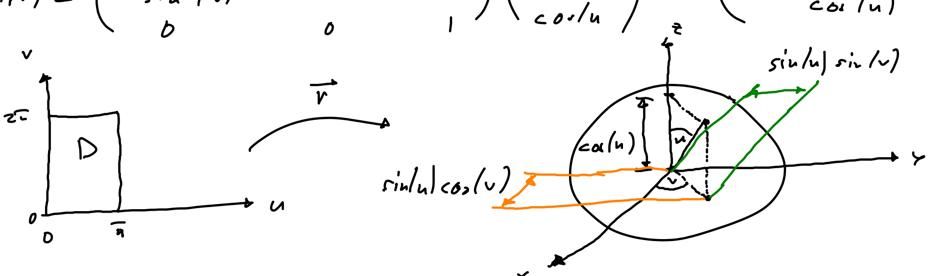
Consider the curve in x-z-plane:

$$\vec{r} : [0, \pi] \longrightarrow \mathbb{R}^3$$
$$u \longmapsto \vec{r}(u) = \begin{pmatrix} \sin(u) \\ 0 \\ \cos(u) \end{pmatrix}$$



rotation: $(v \in [0, 2\pi])$

$$\vec{r}(u,v) = \begin{pmatrix} \cos(v) & -\sin(v) & 0 \\ \sin(v) & \cos(v) & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \sin(u) \\ 0 \\ \cos(u) \end{pmatrix} = \begin{pmatrix} \sin(u)\cos(v) \\ \sin(u)\sin(v) \\ \cos(u) \end{pmatrix}$$



Rem.:

$$\vec{r}(\theta, \varphi) = \begin{pmatrix} \sin(\theta)\cos(\varphi) \\ \sin(\theta)\sin(\varphi) \\ \cos(\theta) \end{pmatrix}$$

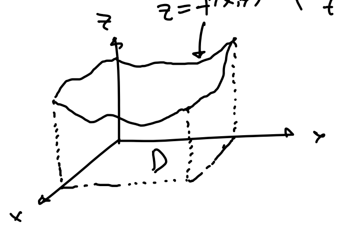
θ : polar angle

φ : azimuthal angle

(v) <u>Graphs</u>: Let $f : \mathbb{R}^2 \longrightarrow \mathbb{R}$

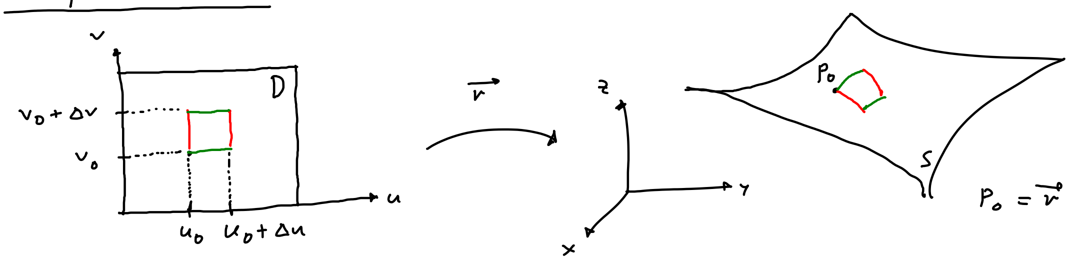
$(x,y) \longmapsto z = f(x,y)$

Then: $\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$ is a surface: the <u>graph of f</u>.



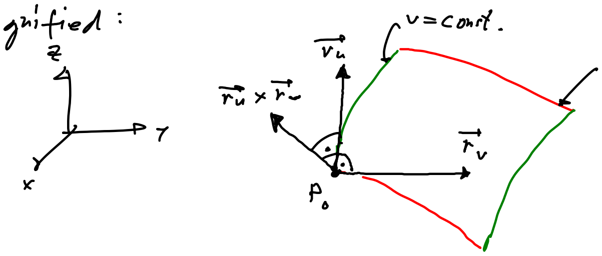
(everything in one picture!)

<u>Surface Area</u>



$P_0 = \vec{r}(u_0, v_0)$

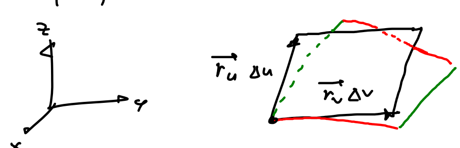
Patch magnified:



$$\vec{r}_u = \frac{\partial \vec{r}}{\partial u}$$

$$\vec{r}_v = \frac{\partial \vec{r}}{\partial v}$$

Approx. of patch element by parallelogram:



$\longrightarrow$ Area of patch element:

$$\Delta A \approx |\vec{r}_u \Delta u \times \vec{r}_v \Delta v|$$

$$= |\vec{r}_u \times \vec{r}_v| \, \Delta u \, \Delta v$$

$\longrightarrow$ Total area $\approx \sum |\vec{r}_u \times \vec{r}_v| \, \Delta u \, \Delta v$

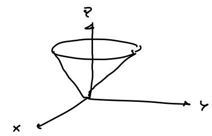
$\longrightarrow$ Area:

$$\boxed{A = \iint_D |\vec{r}_u \times \vec{r}_v| \, du \, dv}$$

Example: Surface area of a cone:

$$\vec{r}(u,v) = \begin{pmatrix} u\cos(v) \\ u\sin(v) \\ u \end{pmatrix} \longrightarrow \vec{r_u} = \frac{\partial \vec{r}}{\partial u} = \begin{pmatrix} \cos(v) \\ \sin(v) \\ 1 \end{pmatrix}$$

$$\vec{r_v} = \frac{\partial \vec{r}}{\partial v} = \begin{pmatrix} -u\sin(v) \\ u\cos(v) \\ 0 \end{pmatrix}$$



$$\longrightarrow |\vec{r_u} \times \vec{r_v}| = \left\| \begin{pmatrix} \cos(v) \\ \sin(v) \\ 1 \end{pmatrix} \times \begin{pmatrix} -u\sin(v) \\ u\cos(v) \\ 0 \end{pmatrix} \right\| = \left\| \begin{pmatrix} -u\cos(v) \\ -u\sin(v) \\ u\cos^2(v) + u\sin^2(v) \end{pmatrix} \right\|$$

$$= u\sqrt{2}$$

$$\longrightarrow A = \iint_D |\vec{r_u} \times \vec{r_v}|\, du\, dv = \int_{v=0}^{2\pi}\int_{u=0}^{1} \sqrt{2}\, u\, du\, dv = \ldots = \sqrt{2}\,\pi$$

Surface integrals

(i) of scalar functions

Def: Let $f: \mathbb{R}^3 \longrightarrow \mathbb{R}$ be a scalar field.
Then the surface integral of f over S is:

$$\iint_S f\, dS = \iint_D f(\vec{r}(u,v))\,|\vec{r_u} \times \vec{r_v}|\, du\, dv$$

Ex: $f = \rho$ with ρ defined only on the surface S and: $\rho(x,y,z)$: mass per unit area.

$$\longrightarrow \iint_S \rho\, dS = \iint_D \rho(\vec{r}(u,v))\,|\vec{r_u} \times \vec{r_v}|\, du\, dv = \text{Mass}.$$

(ii) of vector fields:

Def: Let $\vec{F}: \mathbb{R}^3 \longrightarrow \mathbb{R}^3$ be a vector field.
Then the surface integral of $\vec{F}$ over S is:

$$\iint_S \vec{F} \cdot d\vec{S} = \underbrace{\iint_S \vec{F} \cdot \vec{N}\, dS}_{\text{surf. int. of the scalar field } \vec{F}\cdot\vec{N}},$$

where $\vec{N}$ is the unit normal vector to S.

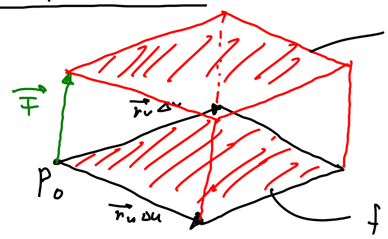
We have: $\vec{N} = \dfrac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}$

$$\longrightarrow \iint_S \vec{F}\cdot d\vec{S} = \iint_S \vec{F}\cdot\vec{N}\, dS$$

$$= \iint_D \vec{F}\cdot \frac{\vec{r}_u \times \vec{r}_v}{|\vec{r}_u \times \vec{r}_v|}\, |\vec{r}_u \times \vec{r}_v|\, du\, dv$$

$$= \iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\, du\, dv$$

Interpretation:



Let $\vec{F}$ be velocity field $|\vec{F}| = 17$ i.e. at P_0, the fluid speed is $17\,m/s$

$\longrightarrow$ Fluid passing through patch in 1s:

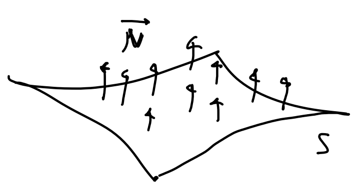
$$\Delta \text{Flux} = \vec{F}\cdot(\vec{r}_u \Delta u \times \vec{r}_v \Delta v)$$
$$= \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,\Delta u\,\Delta v$$

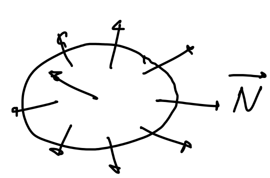
$\longrightarrow$ total flux: $\iint_D \vec{F}\cdot(\vec{r}_u \times \vec{r}_v)\,\Delta u\,\Delta v$

I.e. $\boxed{\iint_S \vec{F}\cdot d\vec{S} = \text{Flux}}$

Remark: Orientation

We call a surface S orientable (or two-sided) if it is possible to define a field $\vec{N}$ of unit normal vectors that varies continuously with position.



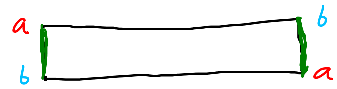


Convention: For closed surface (e.g. sphere) we take $\vec{N}$ to point outward.

Example:

Non orientable surface:

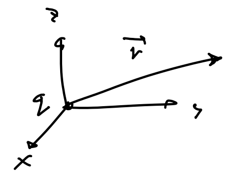
Möbius strip: piece of paper:



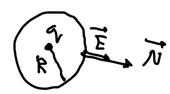
glue green on green with corner a–a, b–b together.

Example: Flux of electric field of point charge q through sphere of radius R:

$$\vec{E}(\vec{r}) = \frac{q}{4\pi\varepsilon_0} \frac{\vec{r}}{|\vec{r}|^3} \quad ; \quad \vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$



$$\text{Flux} = \iint_S \vec{E} \cdot d\vec{S}$$



$$= \iint_S \vec{E} \cdot \vec{N}\, dS$$

$$= \iint_S |\vec{E}|\, dS$$

$$= \iint_S \frac{q}{4\pi\varepsilon_0 R^2}\, dS = \frac{q}{4\pi\varepsilon_0 R^2} \underbrace{\iint_S dS}_{= 4\pi R^2} = \frac{q}{\varepsilon_0}$$

Stokes Theorem

Curl of a vector field

We saw: curl density of 2D-vector field:

$\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix}$

$\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$: measures how much vector field rotates (counter clock wise)

In 3D:

Let: $\vec{F} = \begin{pmatrix} M \\ N \\ 0 \end{pmatrix}$ (no z-component) & $M = M(x,y)$, $N(x,y)$

→ Amount of how this vector field rotates is $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y}$

Axis of rotation is the z-axis.

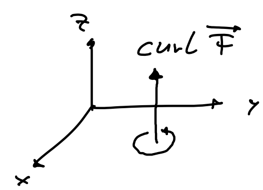
Written as a vector: $\operatorname{curl} \vec{F} = \begin{pmatrix} 0 \\ 0 \\ \dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} \end{pmatrix}$

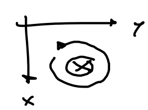
Example: solid rotation in 3D:

$\vec{F} = \dfrac{1}{x^2+y^2} \begin{pmatrix} -y \\ x \\ 0 \end{pmatrix} \longrightarrow \operatorname{curl} \vec{F} = \begin{pmatrix} 0 \\ 0 \\ 2 \end{pmatrix}$

magnitude: how much it rotates

direction: axis of rotation (if you look down on the x-y-plane it rotates counter clock wise if $\dfrac{\partial N}{\partial x} - \dfrac{\partial M}{\partial y} > 0$)





Generalization to arbitrary vector fields:

Definition: Let $\vec{F} = \begin{pmatrix} M \\ N \\ P \end{pmatrix}$

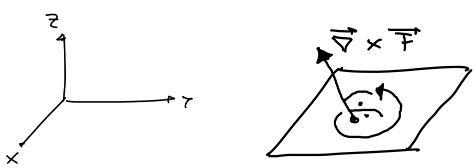
we define the curl of the vector field $\vec{F}$ as:

$$\operatorname{curl} \vec{F} = \begin{pmatrix} \frac{\partial P}{\partial y} - \frac{\partial N}{\partial z} \\ \frac{\partial M}{\partial z} - \frac{\partial P}{\partial x} \\ \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \end{pmatrix}$$

with $\vec{\nabla} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \longrightarrow \underline{\operatorname{curl} \vec{F} = \vec{\nabla} \times \vec{F}}$

<u>interpretation</u> stays the same:

- direction of $\vec{\nabla} \times \vec{F}$: axis of rotation
- magnitude of $\vec{\nabla} \times \vec{F}$: how much it rotates



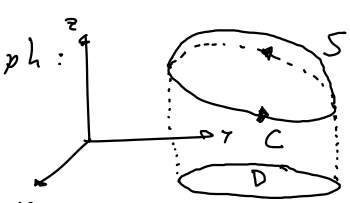
Stokes Theorem

Statement:

$$\boxed{\int_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}} \quad (*)$$

<u>Proof:</u> Restrict to case where S is a graph:

<u>Surface</u> S: $\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ f(x,y) \end{pmatrix}$



$\longrightarrow \quad \vec{r}_x = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{pmatrix} \; ; \quad \vec{r}_y = \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{pmatrix}$

$\longrightarrow \quad \vec{r}_x \times \vec{r}_y = \boxed{\begin{pmatrix} -\frac{\partial f}{\partial x} \\ -\frac{\partial f}{\partial y} \\ 1 \end{pmatrix}} \; ; \quad \nabla \times \vec{F} = \boxed{\begin{pmatrix} \frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \\ \frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \\ \frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \end{pmatrix}} \qquad \vec{F} = \begin{pmatrix} P \\ Q \\ R \end{pmatrix}$

$\longrightarrow$ <u>r.h.s.</u> of $(*)$:

$$\iint_S (\nabla \times \vec{F}) \cdot d\vec{S} = \iint_D \boxed{(\nabla \times \vec{F})} \cdot \boxed{(\vec{r}_x \times \vec{r}_y)}\, dx\, dy$$

$$= \iint_D \left\{ \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\left(- \frac{\partial f}{\partial x} \right) + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\left(- \frac{\partial f}{\partial y} \right) + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) \right\} dx\, dy$$

<u>Curve</u> C : $\quad \vec{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix} \quad ; \; t \in [a, b]$

$\longrightarrow$ <u>l.h.s.</u> of $(*)$:

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b \begin{pmatrix} P(\vec{r}(t)) \\ Q(\dots) \\ R(\dots) \end{pmatrix} \cdot \overbrace{\begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix}}^{\frac{d\vec{r}}{dt}(t)} dt$$

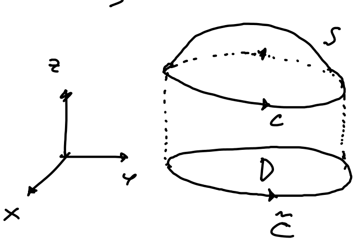
$$= \int_a^b \left(P(\vec{r}(t))\dot{x}(t) + Q(\dots)\dot{y}(t) + R(\dots)\dot{z}(t) \right) dt$$

$$= \int_a^b \left\{ \left(P + R \frac{\partial f}{\partial x} \right) \dot{x} + \left(Q + R \frac{\partial f}{\partial y} \right) \dot{y} \right\} dt$$

$\left[\dot{z}(t) = \frac{dz}{dt}(t) = \frac{\partial f}{\partial x}(x(t), y(t)) \frac{dx}{dt}(t) + \frac{\partial f}{\partial y}(\dots) \frac{dy}{dt}(t) \right]$

$\left[z(t) = f(x(t), y(t)) \quad (\text{Graph!}) \right]$

$$= \int_a^b \begin{pmatrix} P + R \frac{\partial f}{\partial x} \\ Q + R \frac{\partial f}{\partial y} \end{pmatrix} \cdot \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix} dt$$



$$= \int_{\tilde{C}} \begin{pmatrix} P + R \frac{\partial f}{\partial x} \\ Q + R \frac{\partial f}{\partial y} \end{pmatrix} \cdot d\tilde{\vec{r}}$$

$$\underset{\uparrow}{=} \iint_D \left(\frac{\partial}{\partial x}\left(Q + R\frac{\partial f}{\partial y}\right) - \frac{\partial}{\partial y}\left(P + R\frac{\partial f}{\partial x}\right)\right) dx\,dy$$

$\boxed{\text{Green:}}$ $\quad \int_{C} \binom{M}{N}\cdot d\tilde{r} = \iint_D \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$

$Q(x,y,f(x,y))$

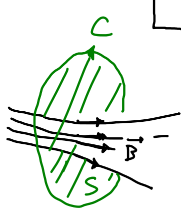
$$\downarrow$$

$$= \iint_D \Bigg(\frac{\partial Q}{\partial x} + \frac{\partial Q}{\partial z}\frac{\partial f}{\partial x} + \left(\frac{\partial R}{\partial x} + \cancel{\frac{\partial R}{\partial z}\frac{\partial f}{\partial x}}\right)\frac{\partial f}{\partial y} + \cancel{R\frac{\partial^2 f}{\partial x \partial y}}$$
$$- \frac{\partial P}{\partial y} - \frac{\partial P}{\partial z}\frac{\partial f}{\partial y} - \left(\frac{\partial R}{\partial y} + \cancel{\frac{\partial R}{\partial z}\frac{\partial f}{\partial y}}\right)\frac{\partial f}{\partial x} - \cancel{R\frac{\partial^2 f}{\partial y \partial x}}\Bigg) dx\,dy$$

$$= \iint_D (\vec{\nabla} \times \vec{F})\cdot d\vec{S} \qquad \square$$

Application: Faraday's law of induction

$$\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$ (One of Maxwell's eqn.)



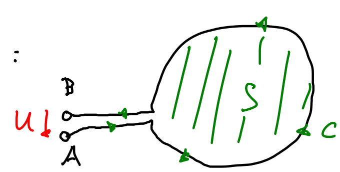
integrating this over surface S yields:

$$\iint_S (\vec{\nabla}\times\vec{E})\cdot d\vec{S} = \iint_S \left(-\frac{\partial \vec{B}}{\partial t}\right)\cdot d\vec{S}$$

$\Vert$ Stokes

$$\int_C \vec{E}\cdot d\vec{r} \qquad = -\frac{d}{dt}\underbrace{\iint_S \vec{B}\cdot d\vec{S}}_{=:\,\Phi(t)\text{ : magnetic flux through } S.}$$

If C is made into:



then, since $\vec{E} = \frac{\vec{F}}{q}$ (force) (charge)

$\longrightarrow \int_C \vec{E}\cdot d\vec{r}$ is work per charge to bring charge from A to B.

since voltage $=$ work per charge

$\longrightarrow \int_C \vec{E} \cdot d\vec{r}$ is the <u>voltage difference</u> from A to B: U

I.e. Faraday's law is:

$$\boxed{U = \int_C \vec{E} \cdot d\vec{r} = -\frac{d}{dt} \iint_S \vec{B} \cdot d\vec{s} = -\frac{d\Phi}{dt}}$$

I.e. induced voltage in loop is equal to the change in magnetic flux!

Other Application:

<u>Ampere's law</u>

$$\vec{\nabla} \times \vec{B} = \mu_0 \Big(\underset{\substack{\uparrow \\ \text{current density}}}{\vec{J}} + \overbrace{\varepsilon_0 \frac{\partial \vec{E}}{\partial t}}^{\text{not due to Ampere}} \Big) \quad \text{(Maxwell eqn.)}$$

Assumption: time indep. $\longrightarrow \frac{\partial \vec{E}}{\partial t} = \vec{0}$

$$\longrightarrow \underline{\vec{\nabla} \times \vec{B} = \mu_0 \vec{J}}$$

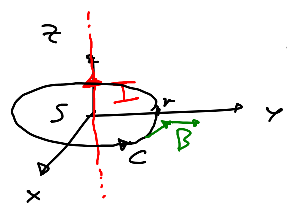
Integrating over S:

$$\iint_S (\vec{\nabla} \times \vec{B}) \cdot d\vec{s} = \underbrace{\iint_S \mu_0 \vec{J} \cdot d\vec{s}}_{\text{current through surface } S} = \mu_0 I$$

Stokes $\parallel$

$$\underbrace{\int_C \vec{B} \cdot d\vec{s}}_{\text{circulation of } \vec{B} \text{ around } C}$$

Example: Wire along z-axis:



Assumption: $\vec{B}$ - field points in circumferential direction

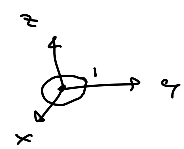
then: $\displaystyle\int_C \vec{B} \cdot d\vec{r} = |\vec{B}|\, 2\pi r \overset{\text{Ampere}}{=} \mu_0 I$

$$\longrightarrow \quad \underline{|\vec{B}| = \frac{\mu_0 I}{2\pi r}}$$

Computational Ex.

$\vec{F} = \begin{pmatrix} z \\ x \\ y \end{pmatrix}$;

We compute work around unit circle in x-y-plane at origin.



We do this in three different ways:

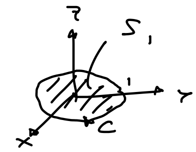
(i) <u>direct:</u> $\displaystyle W = \int_C \vec{F} \cdot d\vec{r} = \int_a^b \vec{F}(\vec{r}(t)) \cdot \frac{d\vec{r}}{dt}(t)\, dt$

$$\vec{r}(t) = \begin{pmatrix} \cos(t) \\ \sin(t) \\ 0 \end{pmatrix} \longrightarrow \; = \int_0^{2\pi} \begin{pmatrix} 0 \\ \cos(t) \\ \sin(t) \end{pmatrix} \cdot \begin{pmatrix} -\sin(t) \\ \cos(t) \\ 0 \end{pmatrix} dt$$

$$= \int_0^{2\pi} \cos^2(t)\, dt = \underline{\pi}$$

(ii) <u>Stokes on unit disk:</u> $\vec{r}(u,v) = \begin{pmatrix} u\cos(v) \\ u\sin(v) \\ 0 \end{pmatrix}$

$(u,v) \in [0,1] \times [0, 2\pi]$



$$W = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

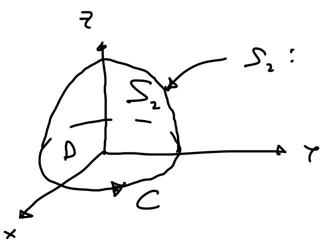
$$= \int_0^{2\pi} \int_0^1 \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix} du\,dv = \int_0^{2\pi} \int_0^1 u\,du\,dv = \underline{\pi}$$

$$\frac{\partial \vec{r}}{\partial u}$$

$$\vec{r}_u \times \vec{r}_v = \begin{pmatrix} \cos(v) \\ \sin(v) \\ 0 \end{pmatrix} \times \begin{pmatrix} -u\sin(v) \\ u\cos(v) \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ u \end{pmatrix}$$

$$\vec{\nabla} \times \vec{F} = \begin{pmatrix} \partial_x \\ \partial_y \\ \partial_z \end{pmatrix} \times \begin{pmatrix} z \\ x \\ y \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

(iii) Stokes on paraboloid



$S_2: z = 1 - x^2 - y^2$

$$\vec{r}(x,y) = \begin{pmatrix} x \\ y \\ 1 - x^2 - y^2 \end{pmatrix}$$

$$\vec{r}_x \times \vec{r}_y = \begin{pmatrix} 1 \\ 0 \\ -2x \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ -2y \end{pmatrix} = \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix}$$

$$W = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

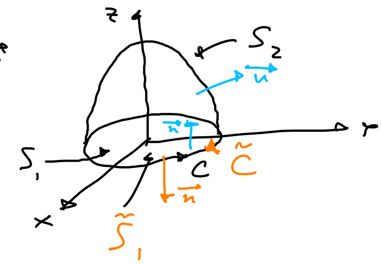
$$= \iint_D \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 2x \\ 2y \\ 1 \end{pmatrix} dx\,dy$$

$$= \iint_D (2x + 2y + 1)\,dx\,dy$$

$$= \int_{\varphi=0}^{2\pi} \int_{r=0}^{1} \left(2r\cos(\varphi) + 2r\sin(\varphi) + 1\right) r\,dr\,d\varphi$$

$$= \int_{\varphi=0}^{2\pi} \left(\frac{2}{3} \underbrace{\left(\cos(\varphi) + \sin(\varphi)\right)}_{\longrightarrow 0} + \frac{1}{2} \right) d\varphi = \underline{\pi}$$

The previous computation tells us:



$$\iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = \int_C \vec{F} \cdot d\vec{r} = \iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

I.e. with $\vec{G} = \vec{\nabla} \times \vec{F}$, then:

$$\iint_{S_1} \vec{G} \cdot d\vec{S} = \iint_{S_2} \vec{G} \cdot d\vec{S}$$

We have: $$\iint_{\tilde{S}_1} \vec{G} \cdot d\vec{S} = -\iint_{S_1} \vec{G} \cdot d\vec{S} = -\iint_{S_2} \vec{G} \cdot d\vec{S}$$

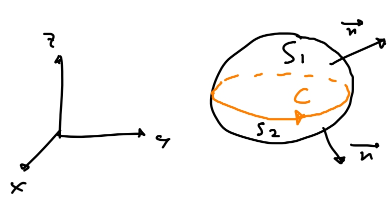
$$\longrightarrow \boxed{\iint_S \vec{G} \cdot d\vec{S} = \iint_{S_2} \vec{G} \cdot d\vec{S} + \iint_{\tilde{S}_1} \vec{G} \cdot d\vec{S} = 0}$$

I.e. $\boxed{\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = 0 \text{ for any closed surface}}$

What we found holds in general:

<u>Theorem:</u> $\boxed{\iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S} = 0 \quad \text{for any closed surface } S}$

<u>Proof:</u>



Idea: Pick closed curve C which lies in S.

Then: $S = S_1 \cup S_2$

We have: (by Stokes)

$$\int_C \vec{F} \cdot d\vec{r} = \iint_{S_1} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

$$\int_C \vec{F} \cdot d\vec{r} = -\iint_{S_2} (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

due to orientation of S_2 and C.

$\longrightarrow$

$$\iint_S (\vec\nabla \times \vec F)\cdot d\vec S = \iint_{S_1} (\vec\nabla \times \vec F)\cdot d\vec S + \iint_{S_2} (\vec\nabla \times \vec F)\cdot dS$$

$$= \int_C \vec F\cdot d\vec r - \int_C \vec F \cdot d\vec r = 0 \qquad \square$$

Rough interpretation: Flux of $\vec G = \vec\nabla \times \vec F$ through closed surface S is zero!
I.e. 'what flows in flows out', like for conserved quantity.
With interpretation of <u>divergence</u> as source strength, we expect:

$$\vec\nabla \cdot \vec G = \underline{\vec\nabla \cdot (\vec\nabla \times \vec F) = 0}.$$

this is true! (see ex.).

Application to gradient fields:

<u>Thm.</u> Let $\vec F$ be defined in a simply connected region.
The following are equivalent:

(i) $\vec F = \vec\nabla \phi$ for some potential ϕ

(ii) $\vec\nabla \times \vec F = \vec 0$

(have seen this in 2D on Oct. 5)

<u>Proof:</u>

<u>(i) $\Rightarrow$ (ii):</u> Computation: $\vec\nabla \times \vec F = \vec\nabla \times \vec\nabla\phi = \vec 0$
(see ex.)

<u>(ii) $\Rightarrow$ (i):</u> $\vec\nabla \times \vec F = \vec 0 \longrightarrow \int_C \vec F \cdot d\vec r = \iint_S \underbrace{(\vec\nabla \times \vec F)}_{=\vec 0} \cdot d\vec S = 0$

C: closed loop

S: surface whose boundary is C.

fund. thm. of line int.

$\longrightarrow$ there exists potential ϕ, s.t. $\vec F = \vec\nabla\phi$.

$\square$

Application:

<u>Electrostatics:</u> Maxwell's eqns: $\vec{\nabla} \cdot \vec{E} = \frac{\rho}{\varepsilon_0}$ (Gauss) (1)

$\vec{\nabla} \times \vec{E} = -\frac{\partial \vec{B}}{\partial t}$ (Faraday) (2)

If static: $\frac{\partial \vec{B}}{\partial t} = \vec{0}$

(2) $\longrightarrow \vec{\nabla} \times \vec{E} = \vec{0} \longrightarrow \underline{\vec{E} = \vec{\nabla}\phi}$

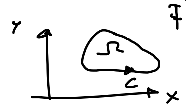
this in (1) $\longrightarrow \vec{\nabla} \cdot \vec{\nabla}\phi = \frac{\rho}{\varepsilon_0}$

I.e. $\underline{\Delta \phi = \frac{\rho}{\varepsilon_0}}$ (Poisson's eqn.).

Divergence Theorem

Overview:

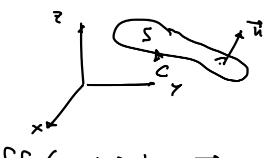
Green, tang.



$\vec{F} = \begin{pmatrix} M \\ N \end{pmatrix}$

$$\int_C \vec{F} \cdot d\vec{r} = \iint_\Omega \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) dA \longrightarrow$$

Stokes:



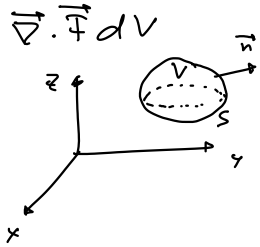
$$\int_C \vec{F} \cdot d\vec{r} = \iint_S (\vec{\nabla} \times \vec{F}) \cdot d\vec{S}$$

Green, normal:

$$\int_C \vec{F} \cdot d\vec{N} = \iint_\Omega \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} \right)}_{= \vec{\nabla} \cdot \vec{F}} dA \longrightarrow$$

Divergence thm. (Gauss)

$$\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F} \, dV$$



Divergence thm.

$$\boxed{\iint_S \vec{F} \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F} \, dV}$$

where:

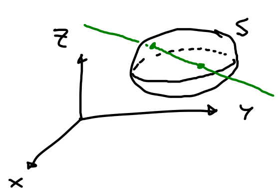
$$\vec{\nabla} \cdot \vec{F} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} \end{pmatrix} \cdot \begin{pmatrix} M \\ N \\ P \end{pmatrix} = \frac{\partial M}{\partial x} + \frac{\partial N}{\partial y} + \frac{\partial P}{\partial z}$$

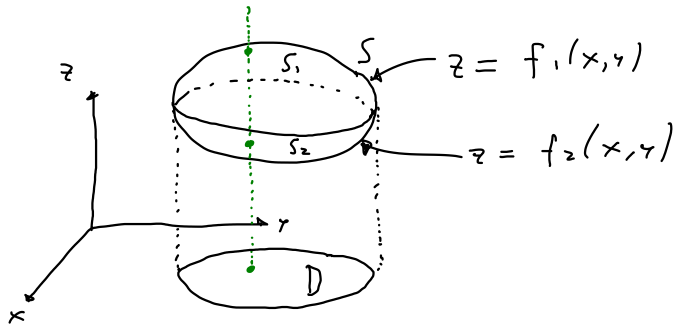
is called the <u>divergence</u> of $\vec{F}$

Proof:

(i) <u>S convex:</u>

<u>Def.:</u> Closed surface S is <u>convex</u> if any straight line meeting it does so in at most two points.





Let $\vec{F} = \begin{pmatrix} M \\ N \\ P \end{pmatrix}$; $\vec{F}_1 = \begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix}$; $\vec{F}_2 = \begin{pmatrix} 0 \\ N \\ 0 \end{pmatrix}$; $\vec{F}_3 = \begin{pmatrix} 0 \\ 0 \\ P \end{pmatrix}$

$\longrightarrow \vec{F} = \sum_{i=1}^{3} \vec{F}_i$

$$\iiint_V \frac{\partial P}{\partial z}\, dV = \underbrace{\int_{x=\cdots}^{\cdots} \int_{y=\cdots}^{\cdots}}_{D} \int_{z=f_2(x,y)}^{f_1(x,y)} \frac{\partial P}{\partial z}(x,y,z)\, dz\, dy\, dx$$

$$= \iint_D \Big(P(x,y,f_1(x,y)) - P(x,y,f_2(x,y)) \Big) dy\, dx \qquad (*)$$

Upper surface : $z = f_1(x,y) \longrightarrow \vec{r}(x,y) = \begin{pmatrix} x \\ y \\ f_1(x,y) \end{pmatrix}$

$$\longrightarrow \vec{r}_x \times \vec{r}_y = \begin{pmatrix} 1 \\ 0 \\ \frac{\partial f_1}{\partial x} \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ \frac{\partial f_1}{\partial y} \end{pmatrix} = \begin{pmatrix} -\frac{\partial f_1}{\partial x} \\ -\frac{\partial f_1}{\partial y} \\ 1 \end{pmatrix}$$

$$\longrightarrow \iint_{S_1} \vec{F}_3 \cdot d\vec{S} = \iint_D \begin{pmatrix} 0 \\ 0 \\ P \end{pmatrix} \cdot \begin{pmatrix} -\frac{\partial f_1}{\partial x} \\ -\frac{\partial f_1}{\partial y} \\ 1 \end{pmatrix} dx\, dy = \iint_D \underset{P(x,y,f_1(x,y))}{\underset{\parallel}{P}}\, dx\, dy$$

$$\longrightarrow \iint_{\substack{S \\ \parallel \\ S_1 \cup S_2}} \vec{F}_3 \cdot d\vec{S} = \iint_D \Big(P(x,y,f_1(x,y)) - P(x,y,f_2(x,y)) \Big) dx\, dy$$

$$= \iiint_V \frac{\partial P}{\partial z}\, dV = \iiint_V \vec{\nabla} \cdot \vec{F}_3\, dV$$

Similarly for $\vec{F}_1 = \begin{pmatrix} M \\ 0 \\ 0 \end{pmatrix}$; $\vec{F}_2 = \begin{pmatrix} 0 \\ N \\ 0 \end{pmatrix}$ but there, integrate w.r.t. x, y first respectively:

$$\iint_S \vec{F}_1 \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F}_1 \, dV$$

$$\iint_S \vec{F}_2 \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F}_2 \, dV$$

$$\longrightarrow \underline{\iint_S \vec{F} \cdot d\vec{S}} = \sum_{i=1}^{3} \iint_S \vec{F}_i \cdot d\vec{S}$$

$$= \sum_{i=1}^{3} \iiint_V \vec{\nabla} \cdot \vec{F}_i \, dV = \underline{\iiint_V \vec{\nabla} \cdot \vec{F} \, dV}$$

$\downarrow$ div. thm. for S convex.

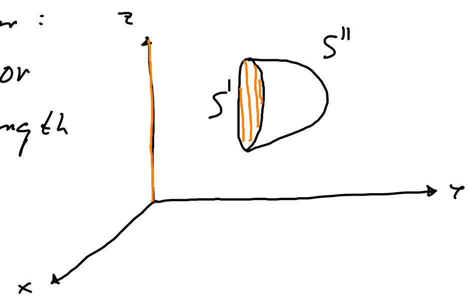
(ii) <u>S semi-convex</u>

<u>Def:</u>

S is called <u>semi-convex</u>, if x-y-z-axes can be chosen, s.t. any line parallel to one of the axes and meeting S either:

(i) does so in at most two points or

(ii) has a portion of finite length in common with S.



Argument as above: $\iint_{S''} \vec{F}_3 \cdot d\vec{S} = \iiint_V \vec{\nabla} \cdot \vec{F}_3 \, dV$

But: $\iint_{S'} \vec{F}_3 \cdot d\vec{S} = 0$ (because: $\vec{F}_3 \cdot \underbrace{(\vec{r}_u \times \vec{r}_v)}_{\text{perpendicular to}} = 0$) 

$$\longrightarrow \iint_S \vec{F}_3 \cdot d\vec{S} = \underbrace{\iint_{S'} \cdots}_{=0} + \iint_{S''} \cdots = \iiint_V \vec{\nabla} \cdot \vec{F}_3 \, dV$$

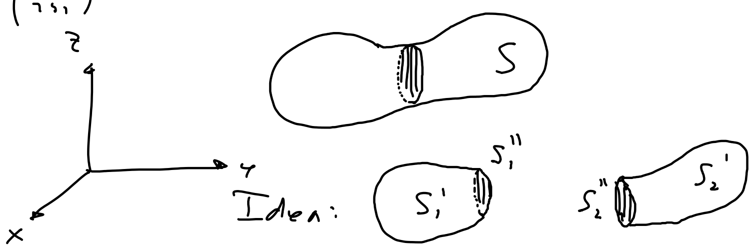
Other directions need no modification.

$$\longrightarrow \iint_S \vec{F}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{F}\,dV$$

$\downarrow$ S semi-convex.

(iii)

not convex or semi-convex?



Idea:

both semi-convex!

$S_1 = S_1' \cup S_1''$

$S_2 = S_2' \cup S_2''$

$S = S_1' \cup S_2'$

We have:

$$\boxed{\iiint_V \vec{\nabla}\cdot\vec{F}\,dV} = \iiint_{V_1} \vec{\nabla}\cdot\vec{F}\,dV + \iiint_{V_2} \vec{\nabla}\cdot\vec{F}\,dV$$

$$= \iint_{S_1} \vec{F}\cdot d\vec{S} + \iint_{S_2} \vec{F}\cdot d\vec{S}$$

$$= \iint_{S_1'} \vec{F}\cdot d\vec{S} + \iint_{S_1''} \vec{F}\cdot d\vec{S} + \iint_{S_2'} \vec{F}\cdot d\vec{S} + \iint_{S_2''} \vec{F}\cdot d\vec{S}$$

$$= \iint_{S_1'} \vec{F}\cdot d\vec{S} + \iint_{S_2'} \vec{F}\cdot d\vec{S} = \boxed{\iint_S \vec{F}\cdot d\vec{S}}$$

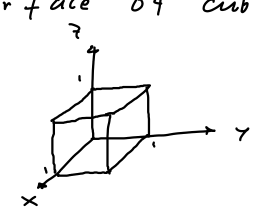
$\iint_{S_1''} \vec{F}\cdot d\vec{S} = -\iint_{S_2''} \vec{F}\cdot d\vec{S}$

$S = S_1' \cup S_2'$

$\square$

Ex:

Flux of $\vec{F} = \begin{pmatrix} xy \\ yz \\ xz \end{pmatrix}$ through surface of cube: $[0,1]^3$



div. thm.

$$\iint_S \vec{F}\cdot d\vec{S} = \iiint_V \vec{\nabla}\cdot\vec{F}\, dV = \iiint_V (x+y+z)\, dV$$

$$\vec{\nabla}\cdot\vec{F} = \frac{\partial}{\partial x}(xy) + \frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial z}(xz) = y + z + x$$

$$= \int_{z=0}^{1}\int_{y=0}^{1}\int_{x=0}^{1}(x+y+z)\,dx\,dy\,dz = \dots = \underline{\frac{3}{2}}$$

Ex: Gauss's law [did this in 2D on Oct. 20]

$$\boxed{\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}}$$ (one of Maxwell's eqn.)

$$\iiint_V \vec{\nabla}\cdot\vec{E}\, dV = \iiint_V \frac{\rho}{\varepsilon_0}\, dV = \frac{1}{\varepsilon_0} Q$$

Q ← total charge in V

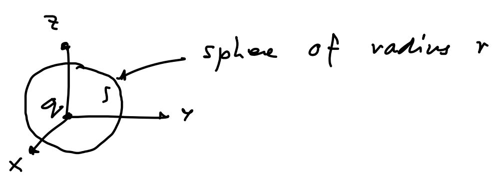
// div. thm.

$$\iint_S \vec{E}\cdot d\vec{S}$$

↖ Flux of electric $\vec{E}$ field through surface S

Gauss's law!

Ex:
Point charge q at origin:



Gauss

$$\frac{q}{\varepsilon_0} = \iiint_V \frac{\rho}{\varepsilon_0}\, dV = \iiint_V \vec{\nabla}\cdot\vec{E}\, dV = \iint_S \vec{E}\cdot d\vec{S} = \iint_S |\vec{E}|\,\vec{N}\cdot d\vec{S}$$

div. thm.

$\vec{E}$ has to point radially (by symmetry)

$$= |\vec{E}| \underbrace{\iint_S \vec{N}\cdot d\vec{S}}_{=4\pi r^2}$$

$$= 4\pi r^2 |\vec{E}| \longrightarrow |\vec{E}| = \frac{q}{4\pi\varepsilon_0 r^2}$$

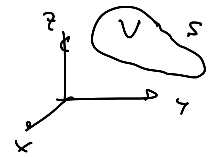
$\vec{E}$ radially $\longrightarrow$ $\underline{\vec{E} = \dfrac{q}{4\pi\varepsilon_0}\dfrac{\vec{r}}{r^3}}$; $r = \sqrt{x^2+y^2+z^2}$, $\vec{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix}$

Force on charge Q: $\vec{F} = Q\vec{E} \longrightarrow$ $\boxed{\begin{array}{ll} \vec{F} = \dfrac{qQ}{4\pi\varepsilon_0}\dfrac{\vec{r}}{r^3} & \\ |\vec{F}| = \dfrac{qQ}{4\pi\varepsilon_0 r^2} & \text{Coulomb's law!} \end{array}}$

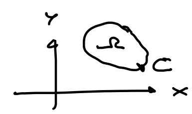
<u>Remark:</u>

Div. thm. works in any dimension!

<u>$n=3$:</u> $$\iiint_V \vec{\nabla}\cdot\vec{F}\,dV = \iint_S \vec{F}\cdot d\vec{S} \quad \text{(Gauss)}$$



<u>$n=2$:</u> $$\iint_\Omega \underbrace{\left(\frac{\partial M}{\partial x} + \frac{\partial N}{\partial y}\right)}_{\vec{\nabla}\cdot\vec{F} \text{ in 2D}} dV = \int_C \vec{F}\cdot d\vec{N} \quad \text{(Green)}$$



<u>$n=1$:</u> $$\int_a^b \frac{df}{dx}\,dx = f(b) - f(a)$$

(Fund. thm. of calculus)