# VECTOR ANALYSIS
# PROBLEM SETS

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## 1. Hand in: 30.9.26, count: 10Pts.

1.1. **Circle rolling on cosine-graph.** A circle with radius $R$ rolls on top of the graph of the cosine function. Find a parametric representation of the path traced out by the center of the circle. Assume that the radius $R$ is sufficiently small, such that the circle travels through the valleys of the cosine graph. Hint: Any graph $y = f(x)$ can be written as a parametric curve by using $x$ as the parameter: $\overrightarrow{r}(x) = (x, f(x))$.

1.2. **Charged particle in homogeneous magnetic field.** The trajectory of a charged particle travelling through a magnetic field $\overrightarrow{B} = (0, 0, B)$ (with $B$ a constant), is given by

$$\overrightarrow{r}(t) = \begin{pmatrix} R\cos(\omega t) \\ R\sin(\omega t) \\ ct \end{pmatrix},$$

where $R$, $\omega$ and $c$ are constants.

(i) Compute the length of the trajectory traced out between $t = 0$ and $t = 5T$, where $T = 2\pi/\omega$.
(ii) Show that the given trajectory $\overrightarrow{r}(t)$ is in agreement with Newton's law of motion $\overrightarrow{F} = m\overrightarrow{a}$, where the acting force $\overrightarrow{F}$ is given by $\overrightarrow{F} = q\overrightarrow{v} \times \overrightarrow{B}$, i.e. given by the Lorentz force.
(iii) Find an expression for $\omega$ in terms of $q$, $B$, $m$.

## 2. Hand in: 7.10.26, count: 10Pts.

2.1. **Arc length of graph.** We consider a curve given by the graph of a function $f : [a, b] \to \mathbb{R}$, $x \mapsto y = f(x)$. Use the arc length formula for curves, as given in class, to show that the arc length of the graph is given by

$$L = \int_a^b \sqrt{1 + (f'(x))^2}dx.$$

2.2. **Gravitational force.** The gravitational potential outside earth is given in good approximation by

$$\phi(x, y, z) = -\frac{GM}{r(x, y, z)}, \qquad \text{where} \qquad r(x, y, z) = \sqrt{x^2 + y^2 + z^2}.$$

Here $G$ is the gravitational constant, $M$ is the mass of the earth and the coordinates have been chosen such that the center of the earth coincides with the origin. The gravitational force acting on a body of mass $m$ is given by $\overrightarrow{F}_g = m\overrightarrow{g}$, where $\overrightarrow{g} = -\overrightarrow{\nabla}\phi$ is the gravitational field. Show that the gravitational force $\overrightarrow{F}_g$ points towards the origin and its magnitude is given by

$$\left|\overrightarrow{F}_g\right| = \frac{GMm}{r^2}.$$

## 3. Hand in: 14.10.26, count: 10Pts.

3.1. **Work for bringing object into orbit.** An object of mass $m$ is brought from the surface of the earth to a height $h$ by following the trajectory given by

$$\begin{aligned}\overrightarrow{r} : [0,1] &\to \mathbb{R}^3 \\ t &\mapsto \overrightarrow{r}(t) = (r_E + th) \begin{pmatrix} \cos(2\pi t) \\ \sin(2\pi t) \\ 0 \end{pmatrix}.\end{aligned}$$

Here $r_E$ is the radius of the earth.

(i) Compute the necessary work using

$$W = \int_C \overrightarrow{F}_g \cdot \overrightarrow{dr},$$

where $\overrightarrow{F}_g$ is the gravitational force as given in the previous problem.

(ii) Compare the result with the one that would be obtained on a purely radial trajectory.

(iii) Explain why the results have to be the same.

3.2. **Path independence and closed loop.** Let $\overrightarrow{F}$ be a vectorfield defined over a domain $\Omega$. Show that the following statementes are equivalent:

(i) $\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$ is path independent,

(ii) $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = 0$ for any closed loop $C$.

Hint: Use $\int_C \overrightarrow{F} \cdot d\overrightarrow{r} = -\int_{C^-} \overrightarrow{F} \cdot d\overrightarrow{r}$, where $C$ and $C^-$ possess the same trace but opposite orientation.

## 4. Hand in: 21.10.26, count: 10Pts.

4.1. **Gradient fields.** For each of the following, determine for what value of the constant $a$ the vector field $\overrightarrow{F}$ will be a gradient field, and for this value, find the corresponding potential function:

(i) $\overrightarrow{F} = \begin{pmatrix} y^2 + 2x \\ axy \end{pmatrix}$, (ii) $\overrightarrow{F} = e^{x+y} \begin{pmatrix} x + a \\ x \end{pmatrix}$.

4.2. **Divergence and curl density.** For each of the following two-dimensional vector fields, draw the vector fields in the domain $[-4,4] \times [-4,4]$ and compute the divergence and the curl density[1]:

(i) $\overrightarrow{F} = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$ (translation)

(ii) $\overrightarrow{F} = \begin{pmatrix} y \\ 0 \end{pmatrix}$ (shear flow)

(iii) $\overrightarrow{F} = \begin{pmatrix} x \\ y \end{pmatrix}$ (uniform expansion)

(iv) $\overrightarrow{F} = \begin{pmatrix} -y \\ x \end{pmatrix}$ (solid rotation)

(v) $\overrightarrow{F} = \frac{1}{x^2+y^2} \begin{pmatrix} x \\ y \end{pmatrix}$ (radial outflow)

[1]The curl density of a vector field $\overrightarrow{F} = \begin{pmatrix} M \\ N \end{pmatrix}$ is given by $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}$.

(vi) $\overrightarrow{F} = \frac{1}{x^2+y^2}\begin{pmatrix} -y \\ x \end{pmatrix}$ (irrotational vortex)

4.3. **Greens Theorem.** Verify both forms of Greens theorem for the vector field

$$\overrightarrow{F} = \begin{pmatrix} x-y \\ x \end{pmatrix}$$

and the region bounded by the unit circle with center at the origin. Hint:

$$\int_0^{2\pi} \cos^2(x)dx = \pi.$$

5. Hand in: 28.10.26, count: 10Pts.

5.1. **Measuring area using Green's theorem.**

(i) Let the vectorfield $\overrightarrow{F}$ be given by

$$\overrightarrow{F} = \frac{1}{2}\begin{pmatrix} -y \\ x \end{pmatrix}$$

Using Green's theorem, show that

$$\int_C \overrightarrow{F} \cdot d\overrightarrow{r}$$

is equal to the area of the region enclosed by the curve $C$.

(ii) Using the above formula for the area of a region in the plane, compute the area of the ellipse enclosed by the curve

$$\overrightarrow{r} : [0, 2\pi] \to \mathbb{R}^2$$
$$t \mapsto \overrightarrow{r}(t) = \begin{pmatrix} a\cos(t) \\ b\sin(t) \end{pmatrix},$$

where $a, b > 0$.

5.2. **Heat equation.** The thermal energy of a substance in a region $\Omega \subset \mathbb{R}^2$ is given by

$$\iint_\Omega \rho c u \, dA,$$

where $\rho$ is the density, $c$ the specific heat and $u$ the temperature of the substance. We assume that $\rho$ and $c$ are constant, while $u = u(t, x, y)$. According to Fourier's law, the heat flux at any point is given by the vector field

$$\overrightarrow{F} = -k\overrightarrow{\nabla} u$$

where $k$ is the constant thermal conductivity of the substance.

(i) The conservation of energy states that the total outflux of heat through the boundary of $\Omega$ is equal to minus the change of thermal energy in $\Omega$. Use this and Green's theorem to derive the heat equation:

$$\frac{\partial u}{\partial t} = \frac{k}{\rho c}\overrightarrow{\nabla} \cdot (\overrightarrow{\nabla} u).$$

(ii) Using the Laplace operator, defined by

$$\Delta f = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2},$$

rewrite the heat equation in the standard form

$$\frac{\partial u}{\partial t} = \frac{k}{\rho c}\Delta u.$$

## 6. Hand in: 4.11.26, count: 10Pts.

6.1. **Parametrization of torus.** We consider the circle in the $x$-$z$-plane with center at $(R, 0, 0)$ and radius $r$ where $0 < r < R$.

(i) Find a parametrization of the circle.
(ii) Rotation of the circle around the $z$-axis generates a surface of revolution. Find a parametrization of this surface.
(iii) The generated surface is called a torus. Create a sketch of the torus.

6.2. **Surface area of graph.** Let a surface be given as the graph of a function $f(x, y)$:

$$\begin{aligned} \vec{r} : D &\to \mathbb{R}^3 \\ (x, y) &\mapsto \vec{r}(x, y) = \begin{pmatrix} x \\ y \\ f(x, y) \end{pmatrix}. \end{aligned}$$

(i) Show that the formula for the surface area

$$A = \iint_D \left| \frac{\partial \vec{r}}{\partial u} \times \frac{\partial \vec{r}}{\partial v} \right| du dv$$

reduces to

$$A = \iint_D \sqrt{1 + \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2}\, dx dy.$$

Remark: This formula is the two dimensional analogue to the formula for the arc length of a graph in problem 2.1.

(ii) Let

$$f(x, y) = \sqrt{1 - x^2 - y^2}, \qquad D = \left\{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\right\}.$$

Compute the surface area. Hint: Compute the resulting double integral in polar coordinates.

## 7. Hand in: 25.11.26, count: 10Pts.

7.1. **Surface area of surfaces of revolution.** Let a curve in the $x$-$z$-plane be given by

$$\begin{aligned} \vec{r} : [a, b] &\to \mathbb{R}^3 \\ u &\mapsto \vec{r}(u) = \begin{pmatrix} f(u) \\ 0 \\ g(u) \end{pmatrix}. \end{aligned}$$

In the following we assume $f(u) \geq 0$, such that $f(u)$ is the distance of the point $\vec{r}(u)$ from the $z$-axis, and we consider the surface of revolution which is generated by rotating this curve around the $z$-axis.

(i) Show that the surface area of the resulting surface of revolution is given by

$$A = 2\pi \int_a^b \sqrt{f(u)^2 \Big(f'(u)^2 + g'(u)^2\Big)}\, du.$$

Here we use the notation $f'(u) = \frac{df}{du}(u)$.

(ii) Use the parametrization of the torus cross section (as found in problem 6.1(i)) and compute the surface area of the torus.

7.2. **Surface integrals of vector fields.** Important: In this problem, the resulting integrals do not have to be computed, but should be given in the form:

$$\int_{x=\dots}^{x=\dots} \int_{y=\dots}^{y=\dots} \dots dydx,$$

where the expression in the integral is a function of $x$ and $y$ (or a similar expression with other parameters in the role of $x$, $y$).

(i) Set up the integral for the flux of the vector field

$$\overrightarrow{F} = \begin{pmatrix} 6z \\ -3y \\ 3 \end{pmatrix}$$

through the part of the plane

$$x + 2y + 2z = 2$$

which lies in the first octant, i.e. in the part of $\mathbb{R}^3$ for which it holds that $x, y, z \geq 0$. Use the orientation for which the normal vector points away from the origin.

(ii) Set up the integral for the flux of the vector field

$$\overrightarrow{F} = \begin{pmatrix} y \\ x \\ z^2 \end{pmatrix}$$

through the lateral surface of the cylinder with radius 5, along the $z$-axis from $z = 0$ to $z = 10$, using the outward pointing normal vector.

Hint: Using the fact that the lateral surface of a cylinder is a surface of revolution, rotate the curve

$$\overrightarrow{r}(u) = \begin{pmatrix} 5 \\ 0 \\ u \end{pmatrix}$$

to find a parametrization.

7.3. **Electric flux of homogeneously charged wire.** The electric field around a homogeneously charged straight wire along the $z$-axis is given by

$$\overrightarrow{E}(x, y, z) = \frac{\lambda}{2\pi\varepsilon_0(x^2 + y^2)} \begin{pmatrix} x \\ y \\ 0 \end{pmatrix},$$

where $\lambda$ is the charge density (charge per unit length of the wire) and $\varepsilon_0$ is the permittivity of vacuum.

(i) Draw the electric field in a plane which is perpendicular to the wire. Assume $\lambda > 0$.

(ii) Show that the flux of the electric field out of a closed coaxial cylinder of height $H$ and radius $R$ (the lateral surface together with the top and the bottom disc) is proportional to $H$ but independent of $R$. Show in particular that the two discs do not contribute to the flux.