LAGRANGIAN MECHANICS LECTURE NOTES February 2024

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Contents

1. Newtonian Mechanics	1
1.1. Single Particle	1
1.2. System of Particles	4
Appendix A. Mathematical Identities	7

The present document contains lecture notes for an elective course at Bern University of Applied Sciences. The notes are meant as a complementary aid for students and no claim to completeness is made.

1. NEWTONIAN MECHANICS

1.1. Single Particle.

1.1.1. Kinematics. We call a body whose extension can be neglected a *particle* and denote by r its position relative to a *reference frame*. This means that the three components of the vector r are cartesian coordinates. We denote them by x, y, z. The *motion* of a particle is then given by the map

$$t \mapsto \boldsymbol{r}(t) = \begin{pmatrix} x(t) \\ y(t) \\ z(t) \end{pmatrix}, \qquad (1)$$

where the parameter t is time. The velocity v(t) and acceleration a(t) are given by

$$\boldsymbol{v}(t) = \dot{\boldsymbol{r}}(t), \qquad \boldsymbol{a}(t) = \dot{\boldsymbol{v}}(t) = \ddot{\boldsymbol{r}}(t), \qquad (2)$$

where we use the notation

$$\dot{\boldsymbol{r}}(t) = \frac{d\boldsymbol{r}}{dt}(t) \tag{3}$$

and the derivative of a vector is given by its component-wise derivative

$$\dot{\boldsymbol{r}}(t) = \begin{pmatrix} \dot{x}(t) \\ \dot{y}(t) \\ \dot{z}(t) \end{pmatrix} . \tag{4}$$

1.1.2. Newtons First Law of Motion. Newtons first law states that:

A body remains at rest or in uniform motion unless acted upon by a force.

In this statement we identify the body with a particle and by being at rest or in uniform motion it is meant that either v = 0 or $v = const \neq 0$, i.e.

$$a = 0. (5)$$

The meaning of the term *force* is only given later through the statement of Newtons second law of motion (see 1.1.3) but it originates from some kind of interaction involving the particle. We call a particle which does not interact a *free* particle and through this notion reformulate Newtons first law of motion as:

There exist reference frames, relative to which a free particle either rests or moves with constant velocity.

These reference frames are called *inertial frames*. I.e. Newtons first law of motion postulates the existence of inertial frames. A good example for an inertial frame is a frame which is at rest relative to the fixed stars.

There exist infinitely many inertial frames, related to each other by coordinate transformations of the form

$$\boldsymbol{r} = \begin{pmatrix} x \\ y \\ z \end{pmatrix} \mapsto \begin{pmatrix} \tilde{x} \\ \tilde{y} \\ \tilde{z} \end{pmatrix} = \tilde{\boldsymbol{r}} \,. \tag{6}$$

Examples:

- (i) $\tilde{\boldsymbol{r}} = \boldsymbol{r} + \boldsymbol{b}$ with \boldsymbol{b} a constant vector is a translation.
- (ii) $\tilde{\boldsymbol{r}} = R\boldsymbol{r}$ with R a constant rotation matrix R is a rotation.
- (iii) $\tilde{r} = r + tc$ with c a constant vector is a motion of the \tilde{x} - \tilde{y} - \tilde{z} -frame relative to the x-y-z-frame with velocity c. Such a transformation is called a *Galilean transformation*.
- (iv) $t \mapsto \tilde{t} = t + d$ with d a constant is a time translation.

Remarks:

- (i) For each of the above examples from $\ddot{r} = 0$ follows that $\ddot{\tilde{r}} = 0$.
- (ii) The transformation

$$\boldsymbol{r} \mapsto \tilde{\boldsymbol{r}} = R\boldsymbol{r} + t\boldsymbol{c} + \boldsymbol{a} \tag{7}$$

$$t \mapsto \tilde{t} = t + d \tag{8}$$

is called a general Galilean transformation.

1.1.3. Newtons Second Law of Motion. Newtons second law of motion states that:

The time rate of change of momentum equals the force acting on the particle.

The corresponding mathematical statement is

$$\dot{\boldsymbol{p}} = \boldsymbol{F} \,, \tag{9}$$

where we define

$$\boldsymbol{p} := m\boldsymbol{v} \tag{10}$$

as the (linear) momentum. Here m is the mass, a (scalar) property of the particle measuring the amount of matter the particle contains. Newtons second law of motion then gives the meaning of the force F acting on the particle, which is a vector.

Remarks:

(i) (9) is a vector equation whose components:

$$F_x = \frac{d}{dt}(mv_x) \qquad F_y = \frac{d}{dt}(mv_y), \qquad F_z = \frac{d}{dt}(mv_z)$$
(11)

are components relative to an inertial frame.

- (ii) In general the force acting on a particle depends on the position \mathbf{r} , the velocity $\dot{\mathbf{r}}$ and the time t, i.e. $\mathbf{F} = \mathbf{F}(\mathbf{r}, \dot{\mathbf{r}}, t)$.
- (iii) If $F(r, \dot{r}, t)$ is known, (9) is a differential equation for the position r(t) of the particle:

$$m\ddot{\boldsymbol{r}}(t) = \boldsymbol{F}\left(\boldsymbol{r}(t), \dot{\boldsymbol{r}}(t), t\right).$$
(12)

1.1.4. Conservation of Linear Momentum. If the force vanishes, then the momentum is a conserved quantity. I.e.

$$F = 0 \qquad \Rightarrow \qquad \dot{p} = 0 \tag{13}$$

Proof. The proof of this statement is a direct consequence of Newtons second law.

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1.1.5. Conservation of Angular Momentum. Let

$$\boldsymbol{L} := \boldsymbol{r} \times \boldsymbol{p}, \qquad \boldsymbol{M} := \boldsymbol{r} \times \boldsymbol{F}$$
 (14)

be the angular momentum and the torque respectively. We have

$$\dot{\boldsymbol{L}} = \frac{d}{dt} (\boldsymbol{r} \times \boldsymbol{p}) = \dot{\boldsymbol{r}} \times \boldsymbol{p} + \boldsymbol{r} \times \dot{\boldsymbol{p}} = \boldsymbol{v} \times \boldsymbol{m} \boldsymbol{v} + \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{m} \underbrace{\boldsymbol{v} \times \boldsymbol{v}}_{=\boldsymbol{0}} + \boldsymbol{r} \times \boldsymbol{F} = \boldsymbol{M}.$$
(15)

I.e. the rate of change of angular momentum equals the torque. The conservation of angular momentum

$$M = 0 \qquad \Rightarrow \qquad \dot{L} = 0 \tag{16}$$

follows analogous to the conservation of linear momentum.

1.1.6. Energy. Let

$$T := \frac{mv^2}{2} \tag{17}$$

be the *kinetic energy*, where v = |v|. Assuming that the mass m is constant, we have

$$\frac{dT}{dt} = \frac{m}{2}\frac{d}{dt}\left(\boldsymbol{v}\cdot\boldsymbol{v}\right) = \frac{m}{2}\left(\dot{\boldsymbol{v}}\cdot\boldsymbol{v} + \boldsymbol{v}\cdot\dot{\boldsymbol{v}}\right) = m\boldsymbol{v}\cdot\dot{\boldsymbol{v}} = \boldsymbol{v}\cdot\boldsymbol{F}.$$
(18)

For an interpretation of this expression we consider a particle at two times t_1 , t_2 . The change in kinetic energy is

$$T_2 - T_1 = \int_{t_1}^{t_2} \frac{dT}{dt} dt = \int_{t_1}^{t_2} \mathbf{F} \cdot \mathbf{v} \, dt \,, \tag{19}$$

where we use the notation $T_i = T(t_i)$ and dropped the arguments in the integral for better readability. We rewrite

$$\boldsymbol{F} \cdot \boldsymbol{v} \, dt = \boldsymbol{F} \cdot \frac{d\boldsymbol{r}}{dt} \, dt = \boldsymbol{F} \cdot d\boldsymbol{r} \,. \tag{20}$$

The corresponding finite expression is

$$\boldsymbol{F} \cdot \Delta \boldsymbol{r} = |\boldsymbol{F}| |\Delta \boldsymbol{r}| \cos(\theta) , \qquad (21)$$

where θ is the angle between the force F and the change in position Δr . I.e. this expression equals the component of F in the direction of Δr times the absolute value of the change in position $|\Delta r|$. This is defined as the *work* done by the force F on the particle while displacing it by Δr . I.e. the difference in kinetic energy equals the work done on the particle.

We call F conservative if there exists a function U(x, y, z) such that $F = -\nabla U$. Remarks:

- (i) U is called *potential* or *potential energy*.
- (ii) $\nabla U = \nabla U(x, y, z)$ and since $\mathbf{F} = -\nabla U$, we have that $\mathbf{F}(\mathbf{r}(t)) = -\nabla U(\mathbf{r}(t))$.
- (iii) The potential U is only fixed up to a constant, i.e. $\nabla U = \nabla (U + C)$ for C a constant. We call E = T + U the *total energy*.

Proposition 1.1. If **F** is conservative, then the total energy E = T + U is conserved.

Proof. From (18) we have

$$\frac{dT}{dt}(t) = \boldsymbol{v}(t) \cdot \boldsymbol{F}(\boldsymbol{r}(t)) = -\boldsymbol{v}(t) \cdot \nabla U(\boldsymbol{r}(t)), \qquad (22)$$

where we wrote out all the argument. Using the chain rule we have

$$\frac{dU}{dt}(t) = \frac{d}{dt}U(\mathbf{r}(t)) = \frac{\partial U}{\partial x}(\mathbf{r}(t))\dot{x}(t) + \frac{\partial U}{\partial y}(\mathbf{r}(t))\dot{y}(t) + \frac{\partial U}{\partial z}(\mathbf{r}(t))\dot{z}(t)$$
$$= \nabla U(\mathbf{r}(t)) \cdot \dot{\mathbf{r}}(t)$$

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$$=\nabla U(\boldsymbol{r}(t)) \cdot \boldsymbol{v}(t), \qquad (23)$$

where for the first equal sign we used the short notation $U(t) = U(\mathbf{r}(t))$. Putting together (22) and (23) yields

$$\frac{dE}{dt}(t) = \frac{dT}{dt}(t) + \frac{dU}{dt}(t) = 0.$$
(24)

1.2. System of Particles. We consider a system of n particles labeled by the index i = 1, ..., n. The equation of motion for the *i*'th particle is Newtons law of motion (9) with the force given as the sum of an external force \mathbf{F}_{ie} and forces due to interactions of the *i*'th with the *j*'th $(j \neq i)$ particle \mathbf{F}_{ij} :

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_{ie} + \sum_{\substack{j=1\\j\neq i}}^n \boldsymbol{F}_{ij} \,, \tag{25}$$

where $\mathbf{p}_i = m_i \mathbf{v}_i = m_i \dot{\mathbf{r}}_i$ and \mathbf{r}_i denotes the position of the *i*'th particle. In the following we restrict our treatment to a system of n = 2 particles, all our results can be generalized to the case of n > 2. For n = 2, the equations of motion for the two particles are

$$\dot{\boldsymbol{p}}_1 = \boldsymbol{F}_{1e} + \boldsymbol{F}_{12} \,, \tag{26}$$

$$\dot{\boldsymbol{p}}_2 = \boldsymbol{F}_{2e} + \boldsymbol{F}_{21} \,. \tag{27}$$

1.2.1. Newtons Third Law of Motion. Newtons third law of motion states that:

The forces that two particles exert on each other are equal in magnitude, of

opposite direction and parallel to the line joining the two particles.

The corresponding mathematical statement is

$$F_{12} = -F_{21}$$
 and $F_{12} \parallel r_2 - r_1$. (28)

1.2.2. Center of Mass Motion. Adding equations (26), (27) we obtain

$$\dot{\boldsymbol{p}}_{1} + \dot{\boldsymbol{p}}_{2} = \boldsymbol{F}_{1e} + \boldsymbol{F}_{2e} + \boldsymbol{F}_{12} + \boldsymbol{F}_{21}$$

= $\boldsymbol{F}_{1e} + \boldsymbol{F}_{2e}$, (29)

where we made use of the first of (28). Defining the *total momentum* P and the *total external* force F_e by

$$\boldsymbol{P} := \boldsymbol{p}_1 + \boldsymbol{p}_2 \,, \tag{30}$$

$$\boldsymbol{F}_e := \boldsymbol{F}_{1e} + \boldsymbol{F}_{2e} \,, \tag{31}$$

respectively, (29) becomes

$$\dot{\boldsymbol{P}} = \boldsymbol{F}_e \,. \tag{32}$$

I.e. if the external forces vanish then the total momentum is conserved, i.e.

$$F_{ie} = \mathbf{0} \qquad \Rightarrow \qquad \dot{P} = \mathbf{0}.$$
 (33)

With the total mass M and the center of mass R defined by

$$M := m_1 + m_2 \,, \tag{34}$$

$$\boldsymbol{R} := \frac{m_1 \boldsymbol{r}_1 + m_2 \boldsymbol{r}_2}{M} \,, \tag{35}$$

respectively, (32) becomes

$$M\ddot{\boldsymbol{R}} = \boldsymbol{F}_{e} \,, \tag{36}$$

the equation of motion for a body of mass M and position \mathbf{R} under the influence of a force \mathbf{F}_e . If $\mathbf{F}_e = \mathbf{0}$, the center of mass follows a uniform motion, i.e.

4

$$\ddot{\boldsymbol{R}} = \boldsymbol{0} \,. \tag{37}$$

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1.2.3. Angular Momentum. By taking the cross product of (26), (27) with r_1 , r_2 , respectively and adding the resulting equations, it can be shown that (see problem 1.2)

$$\dot{L} = M , \qquad (38)$$

where the total angular momentum L and the total torque M are given as

$$\boldsymbol{L} := \boldsymbol{r}_1 \times \boldsymbol{p}_1 + \boldsymbol{r}_1 \times \boldsymbol{p}_1 \,, \tag{39}$$

$$\boldsymbol{M} := \boldsymbol{r}_1 \times \boldsymbol{F}_{1e} + \boldsymbol{r}_2 \times \boldsymbol{F}_{2e} \,. \tag{40}$$

I.e. the rate of change of total angular moment equals the total torque. In particular, if the torque vanishes, the total angular momentum is conserved.

1.2.4. Energy. The kinetic energy of the two particles is given by

$$T = \frac{m_1 v_1^2}{2} + \frac{m_2 v_2^2}{2} \,. \tag{41}$$

We have

$$\frac{dT}{dt} = m_1 \boldsymbol{v}_1 \cdot \dot{\boldsymbol{v}}_1 + m_2 \boldsymbol{v}_2 \cdot \dot{\boldsymbol{v}}_2$$
$$= \boldsymbol{v}_1 \cdot \boldsymbol{F}_1 + \boldsymbol{v}_2 \cdot \boldsymbol{F}_2, \qquad (42)$$

where

$$\boldsymbol{F}_i = \boldsymbol{F}_{ie} + \boldsymbol{F}_{ij} \,. \tag{43}$$

Proposition 1.2. If the interaction forces between particles depend only on the distance between the particles, then they are conservative.

Proof. By the assumption of the proposition, the interaction forces are of of the form

$$F_{12} = f e_{12}, \qquad F_{21} = f e_{21},$$
(44)

where f is a function of one variable (the distance between the particles) and

$$e_{12} = \frac{r_2 - r_1}{|r_2 - r_1|}, \qquad e_{21} = -e_{12}.$$
 (45)

We define the potential U of the system as

$$U(\mathbf{r}_{1}, \mathbf{r}_{2}) := u(|\mathbf{r}_{1} - \mathbf{r}_{2}|), \quad \text{where} \quad u(s) := \int_{s_{0}}^{s} f(\rho) d\rho.$$
(46)

We then have by the chain rule

$$\frac{\partial U}{\partial x_1}(\mathbf{r}_1, \mathbf{r}_2) = \frac{du}{ds}(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{\partial}{\partial x_1}|\mathbf{r}_1 - \mathbf{r}_2|$$
$$= f(|\mathbf{r}_1 - \mathbf{r}_2|)\frac{x_1 - x_2}{|\mathbf{r}_1 - \mathbf{r}_2|}, \qquad (47)$$

where we used

$$\frac{\partial}{\partial x}\sqrt{x^2 + y^2 + z^2} = \frac{x}{\sqrt{x^2 + y^2 + z^2}}.$$
(48)

It follows that

$$\nabla_{1}U(\mathbf{r}_{1},\mathbf{r}_{2}) = f(|\mathbf{r}_{1}-\mathbf{r}_{2}|)\frac{\mathbf{r}_{1}-\mathbf{r}_{2}}{|\mathbf{r}_{1}-\mathbf{r}_{2}|}$$

= $-f(|\mathbf{r}_{1}-\mathbf{r}_{2}|)\mathbf{e}_{12}$
= $-\mathbf{F}_{12}(\mathbf{r}_{1},\mathbf{r}_{2}),$ (49)

where we used the notation

$$\nabla_1 = \begin{pmatrix} \frac{\partial}{\partial x_1} \\ \frac{\partial}{\partial y_1} \\ \frac{\partial}{\partial z_1} \end{pmatrix}.$$
 (50)

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Analogously we find

$$\nabla_2 U(\boldsymbol{r}_1, \boldsymbol{r}_2) = -\boldsymbol{F}_{21}(\boldsymbol{r}_1, \boldsymbol{r}_2), \qquad (51)$$

i.e. the forces are conservative.

Remark: (49), (51) can be written as

$$\nabla U = -\boldsymbol{F},\tag{52}$$

where

$$\nabla = \begin{pmatrix} \nabla_1 \\ \nabla_2 \end{pmatrix}, \qquad \mathbf{F} = \begin{pmatrix} \mathbf{F}_{12} \\ \mathbf{F}_{21} \end{pmatrix}. \tag{53}$$

We have

Proposition 1.3. If the interaction forces between particles depend only on the distance between them and in addition the external forces are conservative, i.e.

$$\boldsymbol{F}_{ie} = -\nabla_i U_e \,, \tag{54}$$

then the total energy of the system

$$E = T + U + U_e \tag{55}$$

is conserved.

Proof. Using

$$\frac{d}{dt}m_i v_i^2 = 2m_i \boldsymbol{v}_i \cdot \dot{\boldsymbol{v}}_i, \qquad \nabla_i U = -\boldsymbol{F}_{ij}, \qquad \nabla_i U_e = -\boldsymbol{F}_{ie}, \qquad (56)$$

together with the equations of motion (26), (27), we obtain

$$\frac{dT}{dt} = \boldsymbol{v}_1 \cdot (m_1 \dot{\boldsymbol{v}}_1 - \boldsymbol{F}_{12} - \boldsymbol{F}_{1e}) + \boldsymbol{v}_2 \cdot (m_2 \dot{\boldsymbol{v}}_2 - \boldsymbol{F}_{21} - \boldsymbol{F}_{2e}) = 0.$$
 (57)

$$\square$$

Appendix A. Mathematical Identities

Product rule for functions of one variable g(t), g(t) and vector valued functions of one variable a(t), b(t):

$$\frac{d}{dt}(fg) = \frac{df}{dt}g + f\frac{dg}{dt},$$
(58)

$$\frac{d}{dt}\left(\boldsymbol{a}\cdot\boldsymbol{b}\right) = \frac{d\boldsymbol{a}}{dt}\cdot\boldsymbol{b} + \boldsymbol{a}\cdot\frac{d\boldsymbol{b}}{dt},$$
(59)

$$\frac{d}{dt}\left(\boldsymbol{a}\times\boldsymbol{b}\right) = \frac{d\boldsymbol{a}}{dt}\times\boldsymbol{b} + \boldsymbol{a}\times\frac{d\boldsymbol{b}}{dt},$$
(60)

where we did not write the argument t.

Chain rule for functions of one variable f(t), g(t) and a function of several variables F(x, y, z) composed with a vector valued function of one variable $\mathbf{r}(t)$:

$$\frac{d}{dt}f(g(t)) = f'(g(t))\frac{dg}{dt}(t), \qquad (61)$$

$$\frac{d}{dt}F(\boldsymbol{r}(t)) = \nabla F(\boldsymbol{r}(t)) \cdot \frac{d\boldsymbol{r}}{dt}(t), \qquad (62)$$

where we denote by f' the derivative of f with respect to its argument.