LAGRANGIAN MECHANICS
PROBLEM SETS

LISIBACH ANDRÉ

In the following we use boldface symbols to denote vectors.

1. Hand in: 5.3.24, count: 15Pts.

1.1. **Rotating Reference Frame.** Let $\boldsymbol{r}(t)$ describe the two-dimensional motion of a particle in an inertial frame. Let $\boldsymbol{s}(t)$ describe the same motion but in a reference frame which is related to the inertial frame by $\boldsymbol{r} = R\boldsymbol{s}$, where

$$R = R(t) = \begin{pmatrix} \cos(\omega t) & -\sin(\omega t) \\ \sin(\omega t) & \cos(\omega t) \end{pmatrix}.$$

I.e. $\boldsymbol{s}(t)$ describes the motion of the particle in a reference frame which rotates counter clockwise relative to the inertial frame with angular velocity ω.

(i) The equation of motion in the inertial frame is given by $m\ddot{\boldsymbol{r}} = \boldsymbol{F}$. Show that the equation of motion in the rotating frame is

$$m\ddot{\boldsymbol{s}} = \boldsymbol{K} + \boldsymbol{F}_{cf} + \boldsymbol{F}_{co},$$

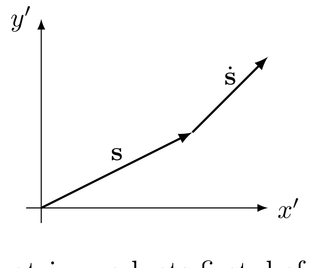
where $\boldsymbol{K} = R^{-1}\boldsymbol{F}$ is the acting force written in components of the rotating frame and

$$\boldsymbol{F}_{cf} = -mR^{-1}\ddot{R}\boldsymbol{s}, \qquad \boldsymbol{F}_{co} = -2mR^{-1}\dot{R}\dot{\boldsymbol{s}}$$

are the centrifugal and Coriolis forces respectively (both being inertial forces).

Hint: Take the time derivative of $\boldsymbol{r} = R\boldsymbol{s}$ twice and multiply the resulting equation with $R^{-1}(t) = R(-t)$ from the left.

(ii) Compute the inertial forces in terms of m, ω, $\boldsymbol{s}$ and $\dot{\boldsymbol{s}}$ and draw these two forces qualitatively in the following situation which shows the location and velocity of the particle in the rotating frame at a certain instant of time:



Hint: Compute the matrix products first, before multiplying with the vectors $\boldsymbol{s}$, $\dot{\boldsymbol{s}}$.

1.2. **Angular Momentum of System of Particles.** Show that the rate of change of the angular momentum of a system of particles equals the torque due to the external forces. I.e. show that

$$\dot{\boldsymbol{L}} = \boldsymbol{M},$$

where $\boldsymbol{L}$ and $\boldsymbol{M}$ are the angular momentum of the system of particles and the torque due to the external forces, respectively, given by

$$\boldsymbol{L} = \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{p}_i, \qquad \boldsymbol{M} = \sum_{i=1}^{n} \boldsymbol{r}_i \times \boldsymbol{F}_{ie}.$$

Hint: Take the cross product of $\boldsymbol{r}_i$ with both sides of the equation of motion for the i'th particle:

$$\dot{\boldsymbol{p}}_i = \boldsymbol{F}_{ie} + \sum_{\substack{j=1 \\ j\neq i}}^{n} \boldsymbol{F}_{ij}$$

and sum over the index i. To convince yourself that the resulting double sum $\sum_{i=1}^{n} \sum_{\substack{j=1 \\ j\neq i}}^{n} \dots$ vanishes, consider first the case $n = 2$ and use Newtons third law.

1.3. **Decomposition of Kinetic Energy.** Show that the kinetic energy T of a system of n particles with masses m_i and velocities $v_i = |\dot{\boldsymbol{r}}_i|$

$$T = \sum_{i=1}^{n} \frac{1}{2} m_i v_i^2$$

can be written as the sum of the kinetic energy of the center of mass motion T_R and the kinetic energy of the relative motion T_r, where

$$T_R = \frac{1}{2} M \dot{\boldsymbol{R}}^2, \qquad T_r = \sum_{i=1}^{n} \frac{1}{2} m_i \dot{\boldsymbol{r}}_{iR}^2.$$

Here M and $\boldsymbol{R}$ are the total mass and the center of mass of the system respectively. Also,

$$\boldsymbol{r}_{iR} = \boldsymbol{r}_i - \boldsymbol{R}$$

is the position of the i'th particle relative to the center of mass.

Hint: Compute the above expression for the kinetic energy T, using $\boldsymbol{r}_i = \boldsymbol{r}_{iR} + \boldsymbol{R}$ and the definition of the center of mass: $\boldsymbol{R} = \sum_i m_i \boldsymbol{r}_i / M$.

2. Hand in: 26.3.24, count: 10Pts.

2.1. **Periodic Motion in Potential Well.** We consider the one dimensional motion of a particle of mass m in a potential well $U(x)$.

(i) Show that the period of motion is given by

$$\tau = \sqrt{2m} \int_{x_1}^{x_2} \frac{dx}{\sqrt{E - U(x)}},$$

where E is the energy and x_1, x_2 are the turning points of the motion.

Hint: Use $E = \frac{1}{2} m \dot{x}^2 + U(x)$, and solve for dt.

(ii) Let now $U(x) = \frac{1}{2} k x^2$, where k is a constant. Show that

$$\tau = 2\pi \sqrt{\frac{m}{k}}.$$

Compare this result with the period found by directly solving Newtons equation of motion $F = m\ddot{x}$ with $F(x) = -U'(x) = -kx$.

Hint: The resulting integral can be solved using the substitution $u = x\sqrt{k/(2E)}$ and

$$-\int \frac{1}{\sqrt{1-u^2}}\, du = \arccos(u) + C.$$

2.2. **Virial Theorem.** The goal of this problem is to show that for periodic, one dimensional motion in a potential $U(x) = ax^n$, with a a constant and $n \in \mathbb{N}$, the virial theorem holds:

$$\overline{T} = \frac{n}{2}\overline{U}.$$

Here $\overline{T}$, $\overline{U}$ are the average values of the kinetic and potential energy respectively over one period of motion. I.e. we use the notation

$$\overline{f} = \frac{1}{\tau}\int_0^\tau f(t)\,dt,$$

where τ is the period of motion.

(i) Considering the quantity

$$G = px,$$

(where p is the momentum) and using Newtons law of motion, show that

$$T = \frac{1}{2}\frac{dG}{dt} - \frac{1}{2}Fx.$$

(ii) Show that

$$Fx = -nU$$

and use this in the above expression for T to show the virial theorem.

(iii) Argue that a stronger version of the theorem is true, namely that $\overline{T} = \frac{n}{2}\overline{U}$ holds even for non periodic motion but the average over one period being replaced by

$$\overline{f} = \lim_{\tau\to\infty}\frac{1}{\tau}\int_0^\tau f(t)\,dt,$$

provided that the position and velocity of the motion stay finite.

3. Hand in: 2.4.24, count: 10Pts.

3.1. **Parabolic and Hyperbolic Orbits.** In the lecture it has been shown that the polar coordinates r, φ of the orbits in Keplers problem satisfy

$$1 + \varepsilon\cos(\varphi) = \frac{p}{r},$$

where

$$\varepsilon = \sqrt{1 + \frac{2El^2}{\mu\alpha}}, \qquad p = \frac{l^2}{\mu\alpha}, \qquad \text{with} \qquad \alpha = Gm_1m_2,$$

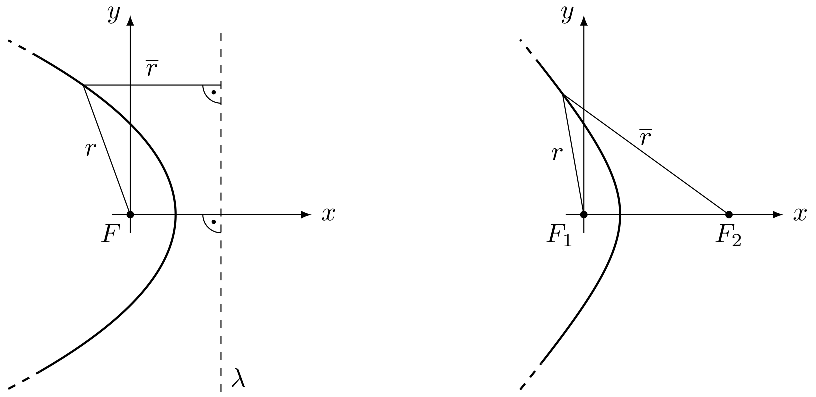
where E is the energy, l the absolute value of the angular momentum, μ the reduced mass, G the gravitational constant and m_1, m_2 are the masses of the sun and the planet respectively.

(i) Any point on a parabola is in equal distance to the focal point F and a line λ. See the figure below. Show that for $E = 0$ the orbit is a parabola with focal point F at the origin and the line λ being perpendicular to the x-axis and intersecting the x-axis at $x = p$.

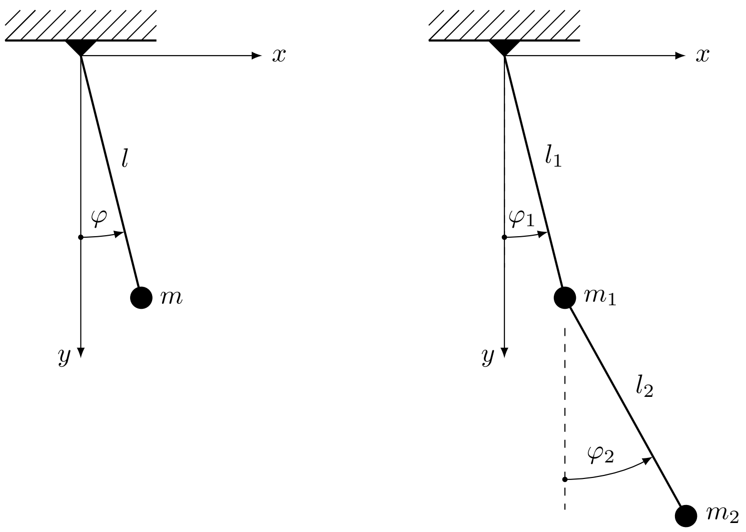
(ii) For any point on a hyperbola the difference of the distances to two focal points F_1, F_2 is constant. See figure below. Show that for $E > 0$ the orbits are of hyperbolic shape with one of the focal points at the origin and the other on the x-axis at $x = 2a\varepsilon$, where $a = p/(\varepsilon^2 - 1)$.

Hint: This can be shown in almost the same way as it has been shown for elliptic orbits in the lecture.

Parabola: $r = \overline{r}$ Hyperbola: $\overline{r} - r = 2a$



3.2. **Lagrangian Function for Pendulum.** In the following we consider the single and double pendulum in a homogeneous gravitational field:



We neglect the masses of the rods l, l_1, l_2 and consider the masses m, m_1, m_2 as point masses. I.e. we consider the pendulums as mathematical pendulums.

(i) Show that the Lagrangian $L = T - U$ for the single pendulum, in terms of the generalized coordinate φ, is given by

$$L = \frac{m}{2} l^2 \dot{\varphi}^2 + mgl\cos(\varphi).$$

Hint: Write down the expressions for $x(t)$, $y(t)$ in terms of $\varphi(t)$ and find from these the kinetic and potential energy

$$T = \frac{m}{2}(\dot{x}^2 + \dot{y}^2), \qquad U = -mgy.$$

(ii) Show that the Lagrangian $L = T - U$ for the double pendulum, in terms of the generalized coordinates φ_1, φ_2, is given by

$$L = \frac{m_1 + m_2}{2} l_1^2 \dot{\varphi}_1^2 + \frac{m_2}{2} l_2^2 \dot{\varphi}_2^2 + m_2 l_1 l_2 \dot{\varphi}_1 \dot{\varphi}_2 \cos(\varphi_1 - \varphi_2)$$
$$+ (m_1 + m_2) g l_1 \cos(\varphi_1) + m_2 g l_2 \cos(\varphi_2).$$

Hint: Show that

$$x_2 = l_1 \sin(\varphi_1) + l_2 \sin(\varphi_2), \qquad y_2 = l_1 \cos(\varphi_1) + l_2 \cos(\varphi_2)$$

and use the derivates of these expressions to write down the kinetic energy of m_2 in terms of φ_1, φ_2. Make use of

$$\cos(a)\cos(b) + \sin(a)\sin(b) = \cos(a - b).$$

4. Hand in: 16.4.24, count: 15Pts.

4.1. **Testing Hamiltons Principle for Freely Falling Body.** We consider a particle of mass $m = 1/2$, freely falling in a homogeneous gravitational field corresponding to an acceleration $g = 2$. Let $x(t)$ denote the position as measured downwards from the initial point and let $x(0) = \dot{x}(0) = 0$.

(i) From Newtonian mechanics we know that the particle falls according to $x(t) = \frac{1}{2}gt^2 = t^2$. In particular $x(1) = 1$. We now consider an alternative path between $t = 0$ and $t = 1$, given by

$$x(t) = t^2 + f(t),$$

where $f(t)$ is an arbitrary function satisfying $f(0) = f(1) = 0$. Show that the action

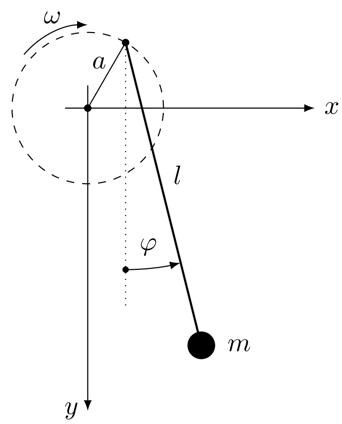
$$S = \int_0^1 L \, dt$$

corresponding to $f(t) \equiv 0$ is minimal.

Hint: Follow the same procedure as discussed in the lecture for the free particle and use integration by parts to take care of the mixed term resulting from $\dot{x}^2$ in the kinetic energy term.

(ii) Use the Euler-Lagrange equation to derive the equation of motion and confirm $x(t)$ as given in (i) using the initial conditions.

4.2. **Pendulum on Wheel.** We consider a mathematical pendulum with mass m and length l whose point of support is attached to a wheel. The radius of the wheel is a, it rotates clockwise with constant angular velocity ω and the point of support is on the positive x-axis at $t = 0$. See the figure.



(i) Find the positions x, y and the velocities $\dot{x}$, $\dot{y}$ of the mass m in terms of the generalized coordinate φ and t.

(ii) Find the expressions for the kinetic and potential energy and from these show that the Lagrangian L is given by

$$L = \frac{m}{2}\Big(a^2\omega^2 + l^2\dot{\varphi}^2 - 2al\omega\dot{\varphi}\sin(\omega t + \varphi)\Big) + mg\Big(a\sin(\omega t) + l\cos(\varphi)\Big).$$

Hint: Use

$$\sin(a)\cos(b) + \cos(a)\sin(b) = \sin(a + b).$$

(iii) Use the Euler-Lagrange equation to show that φ satisfies the equation of motion

$$\ddot{\varphi} = \frac{\omega^2 a}{l}\cos(\omega t + \varphi) - \frac{g}{l}\sin(\varphi).$$

4.3. **Equivalent Lagrangian Functions.** Let $\overline{L}(q, \dot{q}, t)$ and $L(q, \dot{q}, t)$ be two Lagrangian functions differing by the time derivative of some function $F(q, t)$, i.e.

$$\overline{L}(q, \dot{q}, t) = L(q, \dot{q}, t) + \frac{d}{dt} F(q, t).$$

Show that the equations of motion corresponding to $\overline{L}$ and L are the same.

Hint: Consider the action

$$\overline{S} = \int_{t_1}^{t_2} \overline{L}(q, \dot{q}, t) dt$$

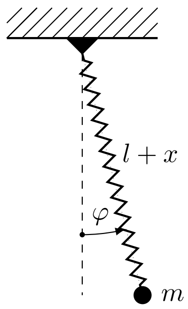
using the fundamental theorem of calculus and examine the effect of the additional terms when deriving the Euler-Lagrange equations, i.e. when computing

$$\frac{d}{d\varepsilon} \overline{S}[q + \varepsilon \tilde{q}] \Big|_{\varepsilon=0},$$

where $\tilde{q}$ is an arbitrary function with $\tilde{q}(t_1) = \tilde{q}(t_2) = 0$.

5. Hand in: 30.4.24, count: 15Pts.

5.1. **Spring Pendulum.** We consider a pendulum whose rod is made up of a spring with spring constant k and equilibrium length l. We describe the length of the spring by $l + x(t)$.



(i) Show that the Lagrangian is given by

$$L = \frac{m}{2}\left(\dot{x}^2 + (l + x)^2 \dot{\varphi}^2\right) + mg(l + x)\cos(\varphi) - \frac{k}{2} x^2.$$

(ii) Show that the equations of motion are given by

$$\ddot{x} = (l + x)\dot{\varphi}^2 - \frac{k}{m} x + g\cos(\varphi),$$

$$\ddot{\varphi} = -\frac{1}{l + x}\left(2\dot{\varphi}\dot{x} + g\sin(\varphi)\right).$$

5.2. **Index of Refraction.** We consider a slab of material in which the speed of light is proportional to the height above the base of the slab. The goal of the present problem is to show that the light moves in circular arcs in this material, assuming that the light takes the path of least time between two points (Fermats principle).

We put the slab such that the base of the slab corresponds to the $y = 0$ plane. It then suffices to consider two dimensional paths described by $y(x)$.

(i) Using $v = \frac{ds}{dt}$ for the speed of light, together with $ds = \sqrt{1 + y'^2} dx$ and the fact that $v = ky$ (for some $k = \text{const.}$), show that the travel time T for the light between x_1 and x_2 satisfies

$$Tk = \int_{x_1}^{x_2} L(y, y') dx, \qquad \text{where} \qquad L(y, y') = \frac{\sqrt{1 + y'^2}}{y}.$$

(ii) Show that the corresponding Euler Lagrange equation is

$$\frac{d}{dx}(yy') = -1.$$

Hint: Intermediate results:

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{1}{y\sqrt{1+y'^2}}\left(y'' - \frac{y'^2}{y} - \frac{y'^2 y''}{1+y'^2}\right), \qquad \frac{\partial L}{\partial y} = -\frac{\sqrt{1+y'^2}}{y^2}.$$

(iii) Integrating the Euler Lagrange equation once yields

$$yy' = -x + C.$$

Show that the solution of this equation is given by

$$(x-A)^2 + y^2 = B^2$$

for some constants A, B. I.e. the solutions are circles with center at the base of the slab.

Hint: Use separation of variables to integrate the equation and completing the square to obtain the equation for the circle.

5.3. **Moving Pendulum.** In the lecture it has been shown that the Lagrangian L of the moving pendulum is given by

$$L = \frac{M+m}{2}\dot{x}^2 + ml\dot{x}\dot{\varphi}\cos(\varphi) + \frac{m}{2}l^2\dot{\varphi}^2 + mgl\cos(\varphi).$$

Here l is the rod length, M the upper mass (moving horizontally) and m the lower mass (attached to the pendulum). x (horizontal position of upper mass) and φ (angle relative to vertical position) are the generalized coordinates.

(i) Compute the canonical momentum p_x and argue why it is a conserved quantity.

(ii) Compute the Hamiltonian function

$$H = \sum_{i=1}^{2} \frac{\partial L}{\partial \dot{q}_i}\dot{q}_i - L$$

and compare the result with the mechanical energy of the system.

(iii) Recall from the lecture that the position of the mass m attached to the pendulum in terms of the generalized coordinates is given by

$$X = x + l\sin(\varphi), \qquad Y = l\cos(\varphi).$$

Show that for $p_x = 0$ (see (i)) the mass m moves on an ellipse.

Hint: $p_x = 0$ implies

$$(M+m)x + ml\sin(\varphi) = C = \text{const.},$$

where, by an appropriate choice of origin for the x-coordinate, we can set $C = 0$. Use this to show that

$$\begin{pmatrix} X \\ Y \end{pmatrix} = l\begin{pmatrix} \frac{M}{m+M}\sin(\varphi) \\ \cos(\varphi) \end{pmatrix},$$

i.e. the mass m moves on an ellipse.